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# MATHEMATICS

4 ESO

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# MATHEMATICS 4 ESO

## CONTENTS

# UNIT 5:

# "PERIMETERS, SURFACES AND VOLUMES. SIMILARITIES"



KEY VOCABULARY:

Polygon
Regular
Irregular
Perimeter
Area
Surface
Quadrilateral
Composite
Apothem
Circular
Sector
Segment
Annulus
Polyhedron
Platonic solid
Prism
Cuboid
Convex
Concave
Pyramid
Apex
Surface development
Tetrahedron
Solid of revolution
Axis of revolution
Cylinder
Cone
Sphere
Spherical cap
Spherical wedge
Ungula
Similar polygons
Ratio of similarity
Ratio of proportionality

*In this unit you will review how to:*

- *Calculate the perimeter and the area of a polygon*
- *Calculate the perimeter and the area of a circular shape*
- *Calculate the area of a prism*
- *Calculate the area of a pyramid*
- *Calculate the area of a polyhedron*
- *Calculate the area of a solid of revolution*
- *Calculate the volume of a 3-D shape*

*In this unit you will learn how to:*

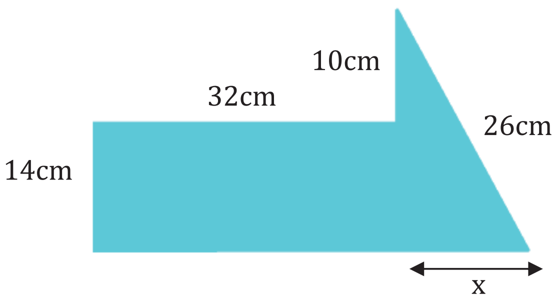
- *Identify similar shapes*
- *Calculate the ratio of similarity of similar shapes*
- *Calculate the ratio of proportionality between areas of similar shapes*
- *Calculate the ratio of proportionality between volumes of similar shapes*

## 1. Polygons. Perimeter of a polygon

A polygon is a closed plane figure that is bounded by three or more straight line segments. There are both regular polygons, which are polygons with equal sides, and irregular polygons which are polygons whose sides have different lengths.

The **perimeter** of a polygon is the sum of the lengths of its sides.

Example: "Calculate the perimeter of the following irregular polygon:"




First of all, we need to calculate the length of the base by using the Pythagoras' theorem:

$$x = \sqrt{26^2 - 24^2} = 10 \text{ cm}$$

So, the base is 42cm long and the perimeter is:

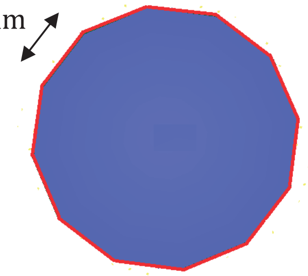
$$P = 14 + 32 + 10 + 26 + 42 = 124\text{cm}$$

In the case of a regular polygon with n sides, we can use the following formula to calculate its **perimeter** ( **P** ) :

$$\boxed{P = n \cdot \ell}$$

where $\ell$ is the length of any side

Example: "Calculate the perimeter of the following regular polygon:"




$P = 12 \cdot 4 = 48$ dm

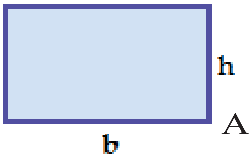
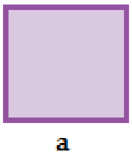
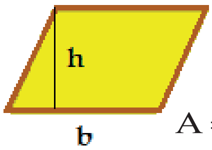
## 2. Area of a polygon

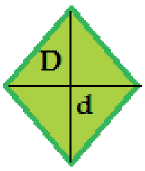
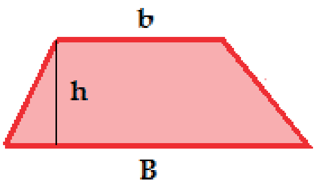
The mathematical term "Area" or "Surface" of a two-dimensional shape can be defined as the amount of space taken up by that shape.

As you probably know, a quadrilateral is a 4-sided polygon.

Here there are formulas of the areas of some quadrilaterals.

**ONE TIP:** Remember those formulae

| RECTANGLE | SQUARE | RHOMBOID |
|---|---|---|
|   $A = b \cdot h$ |   $A = a \cdot a = a^2$ |   $A = b \cdot h$ |
| 4 right interior angles<br>Opposite sides are parallel | Special case of a rectangle whose four sides are equal length | No right angles<br>Opposite sides and angles are equal |

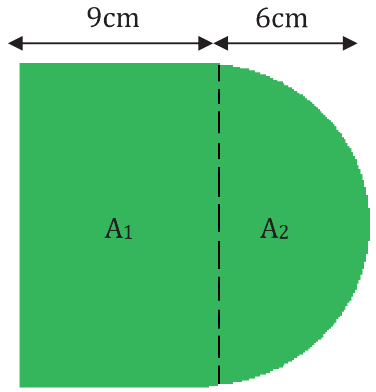
| RHOMBUS | TRAPEZOID |
|---|---|
|   Long diagonal, Short diagonal $A = \frac{D \cdot d}{2}$ |   Long base, Short base $A = \frac{(B + b) \cdot h}{2}$ |
| No right angles<br>Four sides are equal length<br>Opposite sides and angles are equal | No right angles<br>One pair of parallel sides<br>One pair of non parallel sides |

A composite shape is a shape that can be divided into two or more basic shapes whose area can be calculated by using a known formula.

In the case of a composite shape, to calculate its area ( **A** ) we simply calculate the areas of the basics shapes that form it and then we add them together.

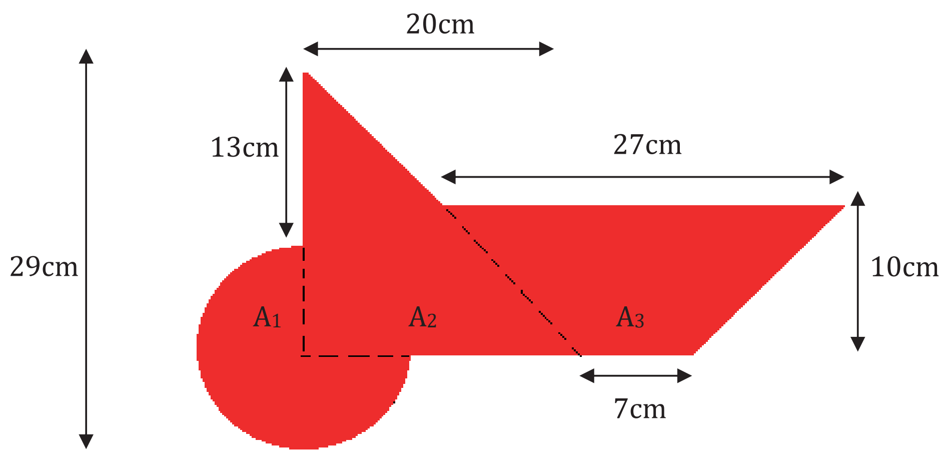
Example: "Calculate the area of the following shapes:"

a)




$$A = A_1 + A_2 = 9 \cdot 12 + \frac{\pi \cdot 6^2}{2} \approx 165{,}55\text{cm}^2$$

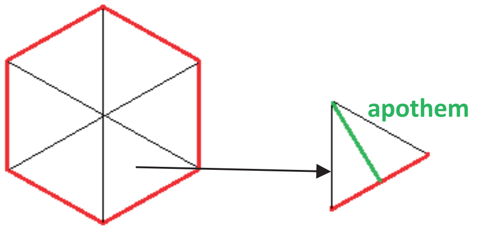
b)




$$A = A_1 + A_2 + A_3 = \frac{3}{4} \cdot \pi \cdot 8^2 + \frac{20 \cdot 21}{2} + \frac{27+7}{2} \cdot 10 \approx 530{,}8\text{cm}^2$$

**In the case of a regular polygon with n sides**, its area ( **A** ) can be calculated as the sum of the areas of the n isosceles triangles that polygon can be divided:

For instance: **a hexagon can be divided into 6 triangles**




Remember also that **a hexagon is the only polygon for which the side and the radius are same length**.

n is the number of triangles

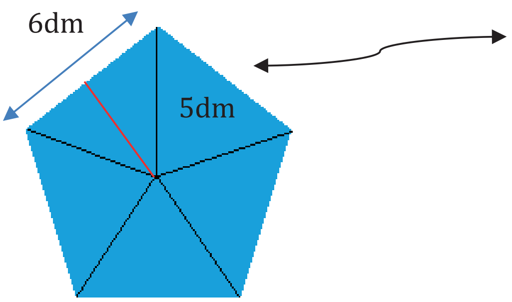
Notice that's the perimeter of the polygon

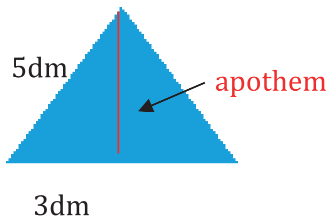
$$\mathbf{A}_{\mathbf{polygon}} = n \cdot A_{triangle} = n \cdot \frac{\mathbf{side} \cdot \mathbf{height}}{2} = \frac{\mathbf{n \cdot side} \cdot \mathbf{apothem}}{2}$$

In conclusion:

$$\boxed{\mathbf{A}_{\mathbf{regular\ polygon}} = \frac{\mathbf{perimeter \cdot apothem}}{\mathbf{2}}}$$

Example: "Calculate the area of the following shape:"



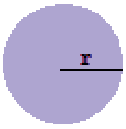
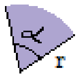

$$\text{apothem} = \sqrt{25-9} = 4 \text{ dm}$$

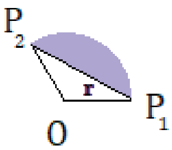
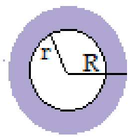
$$P = 5 \cdot 6 = 30 \text{ dm}$$

$$A = \frac{30 \cdot 4}{2} = 60 \text{ dm}^2$$

## 3.Circular shapes.Perimeter and area of some circular shapes

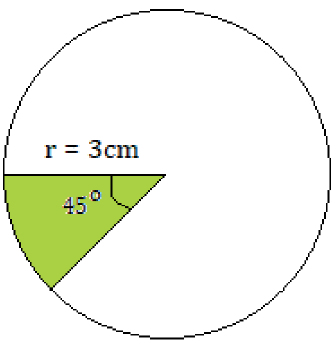
A circular shape is a shape that includes at least one portion of a circle. Let's study the perimeter and the area of some circular shapes:

| CIRCLE | CIRCULAR SECTOR |
|---|---|
|   $P = 2\pi r$ <br> $A = \pi \cdot r^2$ <br> plane surface sorrounded by a circumference |   $P = \frac{2\cdot\pi\cdot r\cdot\alpha}{360} + 2r$ <br> $A = \frac{\pi\cdot r^2\cdot\alpha}{360}$ <br> part of a circle enclosed by two radii |

| CIRCULAR SEGMENT | ANNULUS ( or RING) |
|---|---|
|   $P = \frac{2\cdot\pi\cdot r\cdot\alpha}{360} + P_1P_2$ <br> $A = \frac{\pi r^2\cdot\alpha}{360} - A_{triangle\,OP_1P_2}$ <br> region of a circle enclosed by a chord and its associated arc |   $P = 2\pi R + 2\pi r =$ <br> $= 2\pi(R + r)$ <br> $A = \pi\cdot(R^2 - r^2)$ <br> region of a circle enclosed by two concentric circumferences |

Example: "Calculate the perimeter and the area of the following circular shapes:

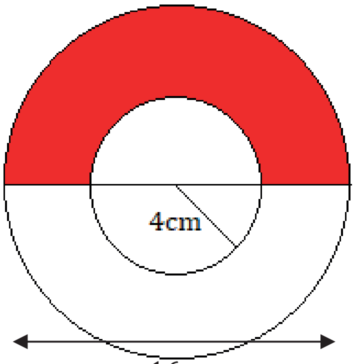
a)




$$P = \frac{2\cdot\pi\cdot 3\cdot 45}{360} + 2\cdot 3 \approx 8{,}36 \text{ cm}$$

$$A = \frac{\pi\cdot 3^2\cdot 45}{360} \approx 3{,}53 \text{ cm}^2$$

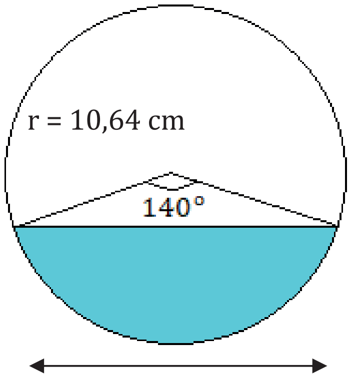
b)




$$P = \frac{1}{2} \cdot 2\pi(8 + 4) + 8 \approx 45{,}7 \text{ cm}$$

$$A = \frac{1}{2} \cdot \pi(8^2 - 4^2) \approx 75{,}4 \text{ cm}^2$$

c)

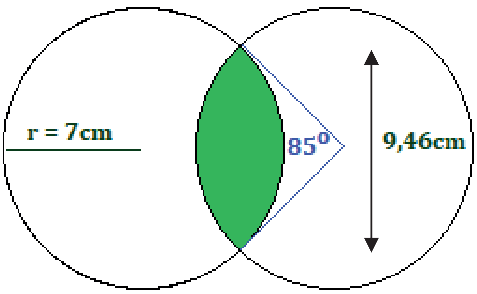



$$P = \frac{2 \cdot \pi \cdot 10{,}64 \cdot 140}{360} + 20 \approx 46 \text{ cm}$$

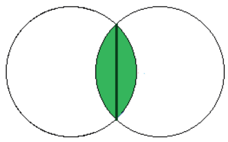
$$A_{\text{triangle}} = \frac{20 \cdot \sqrt{10{,}64^2 - 10^2}}{2} \approx 36{,}35 \text{ cm}^2$$

$$A = \frac{\pi \cdot 10{,}64^2 \cdot 140}{360} - 36{,}35 \approx 101{,}96 \text{cm}^2$$

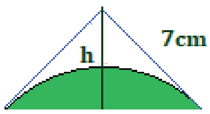
d)




As you can see, it can be considered as double of a circular segment:



So, we only need to apply the formula. First, we calculate the area of the triangle:

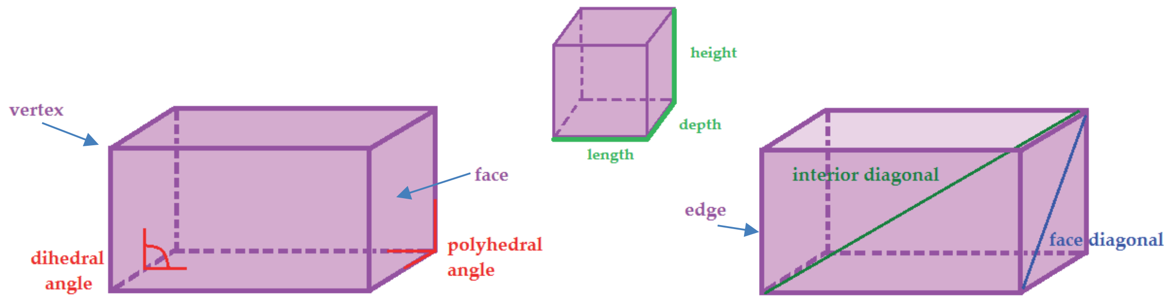



$$A_{\text{triangle}} = \frac{9{,}45 \cdot \sqrt{7^2 - 4{,}725^2}}{2} \approx 24{,}4 \text{ cm}^2$$

Therefore, the final calculation is: $A = 2 \cdot \left[ \frac{\pi \cdot 7^2 \cdot 85}{360} - 24{,}4 \right] \approx 23{,}89 \text{ cm}^2$

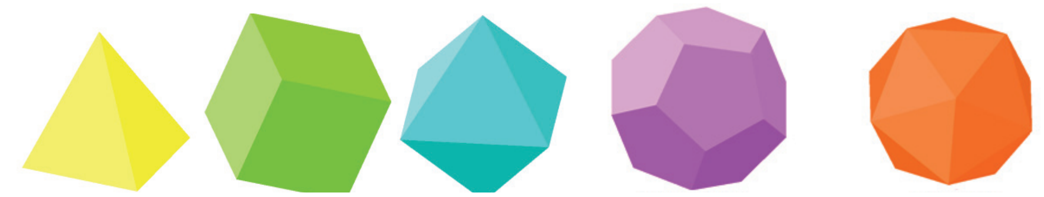
## 4. Polyhedrons. Area of a polyhedron

Remember that, a Polyhedron is a three dimensional shape formed by polygons. It is recommendable to remember also some of the elements of a polyhedron:




It is also recommendable to remember **that there are many types of polyhedrons** , for instance:

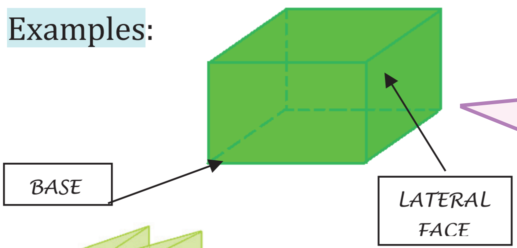
1 **The five** Regular polyhedrons also called the PLATONIC SOLIDS.



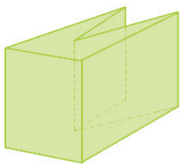

Remember also that a polyhedron is Regular when all its faces are identical regular polygons. Otherwise it is called Irregular.

2 Prisms which are polyhedrons that have two identical parallel faces called bases that identify the prism and the rest of the faces are parallelograms which are called lateral faces.

Examples:

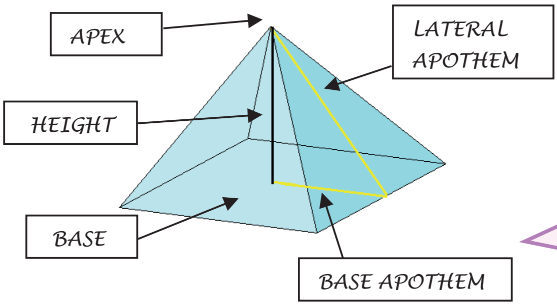



Remember that's a polyhedron called **CUBOID**, which is a Convex (a straight line can only cut its surface at two points) polyhedron bounded by six rectangular faces.



In this case, the polyhedron is Concave because a straight line can cut its surface at more than two points.

3 Pyramids are polyhedrons whose base is a polygon and whose lateral sides are triangles that meet at the top in a point called apex. Let's remember some elements of a pyramid:




Remember that, if their sides are regular polygons:

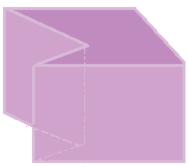
* The lateral apothem of a pyramid is a perpendicular line from the apex of the pyramid to a side of its base.
* The base apothem is a perpendicular line from the center of the base to a side of that base.

We have already revised some of the main types of polyhedrons. Let's remember a formula that states for all convex polyhedrons and for some concave polyhedrons. It is the Euler's formula:

$$F + V = E + 2$$

where **F** is the number of faces of the polyhedron, **V** is the number of its vertices and **E** is the number of its edges.

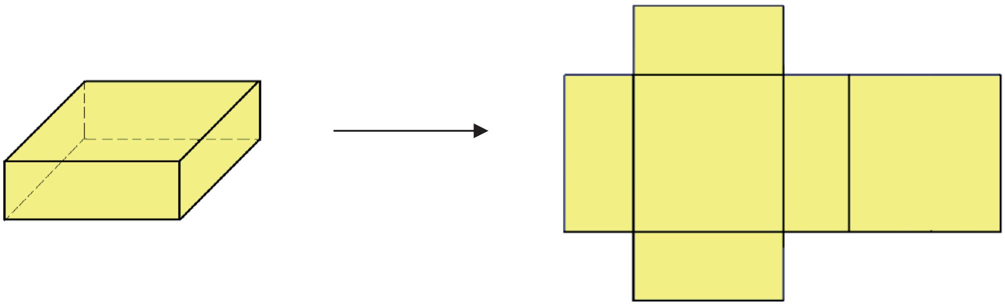
Example: for instance, the Euler's formula states for the following polyhedron despite the fact that it is concave.



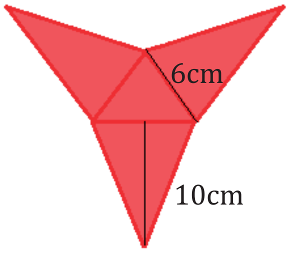
F = 7
V = 10
E = 15

$\longrightarrow$ 7 + 10 = 15 + 2

As our objective is to talk about areas of polyhedrons, at this point, there is a concept that is very useful to remember, the Surface development, which is the plain surface of a solid that has been constructed by bending that plain surface.



Example: “Calculate the area of the following surface development:”




Logically, we need to add the four areas, one corresponding to the equilateral triangle (Ae) and the three corresponding to the isosceles triangles ( Ai ).

$$A_e = \frac{6 \cdot \sqrt{6^2 - 3^2}}{2} \approx 15{,}59 \text{ cm}^2$$

$$A_i = \frac{6 \cdot 10}{2} \approx 30 \text{ cm}^2$$

Therefore, the area of the surface development is: $A \approx 15{,}59 + 3 \cdot 30 = 105{,}59 \text{ cm}^2$

Let’s see some formulas in order to calculate areas of polyhedrons in a more practical way.

## 4.1. Prisms. Area of a prism

The formula of the area of a regular prism stands:

$$A = A_{lateral} + 2 \cdot A_{base}$$

Remember a prism has 2 bases

where $A_{lateral} = n \cdot A_{rectangle} = n \cdot (\text{base}_{rectangle} \cdot \text{height}) = \text{Perimeter of the base} \cdot \text{height}$

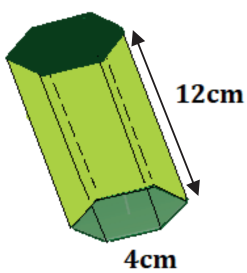
n is the number of sides of the base

In conclusion, the formula of the area of a **regular prism** can also be expressed:

$$A = A_{lateral} + 2 \cdot A_{base} = \text{Perimeter}_{base} \cdot \text{height} + 2 \cdot A_{base}$$

Examples : “ Calculate the area of the following prisms: ”

a)

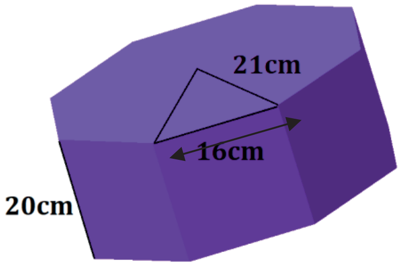



This is a regular, right( in spite of the drawing perspective), hexagonal prism. As it is regular, its bases are regular hexagons and in this case it is easy to calculate the area of the bases.

$$A_{base} = \frac{6 \cdot 4 \cdot \sqrt{4^2 - 2^2}}{2} \approx 41{,}57 \text{ cm}^2$$

Therefore, the area of the prism is: $A \approx 6 \cdot 4 \cdot 12 + 2 \cdot 41{,}57 = 371{,}14 \text{ cm}^2$

b)




This is a regular, right ( in spite of the drawing perspective) octagonal prism. Then again, it is easy to calculate the area of the bases.

$$A_{base} = \frac{8 \cdot 16 \cdot \sqrt{21^2 - 8^2}}{2} \approx 1242{,}66 \text{ cm}^2$$

Therefore, the area of the prism is: $A \approx 8 \cdot 16 \cdot 20 + 2 \cdot 1242{,}66 = 5045{,}32 \text{ cm}^2$

## 4. 2. Pyramids. Area of a pyramid

The formula of the area of a regular pyramid stands:

$$A = A_{lateral} + A_{base}$$

Remember a prism has only one base

where $A_{lateral} = n \cdot A_{triangle} = n \cdot \left( \frac{base_{triangle} \cdot height_{triangle}}{2} \right) = \frac{Perimeter_{base} \cdot apothem}{2}$

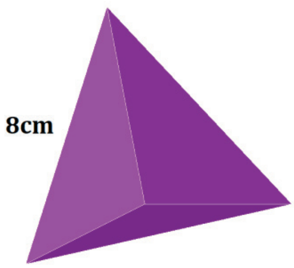
n is the number of sides of the base

Don't forget that the height of the triangle is the apothem of the pyramid

In conclusion, the formula of the area of a **regular pyramid** can also be expressed:

$$A = A_{lateral} + A_{base} = \frac{Perimeter_{base} \cdot apothem}{2} + A_{base}$$

Example : " Calculate the area of the following regular pyramid: "



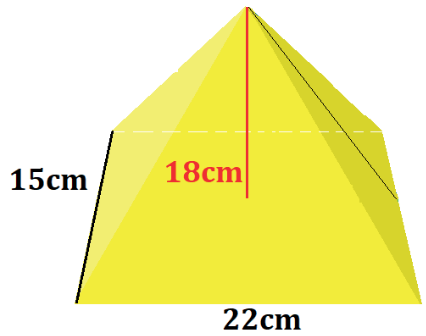

This is a regular pyramid. More precisely, it is a tetrahedron. Let's calculate its area:

$$A = \frac{3 \cdot 8 \cdot \sqrt{8^2 - 4^2}}{2} + \frac{8 \cdot \sqrt{8^2 - 4^2}}{2} \approx 110{,}85 \text{ cm}^2$$

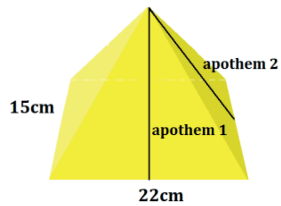
## 4. 3. Area of a 3D-shape

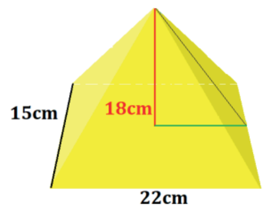
★ Sometimes, we need to calculate the area of an **irregular polyhedron**. In those cases, we need to remember the concept of surface development to make it easier.

Examples : " Calculate the area of the following irregular pyramid: "

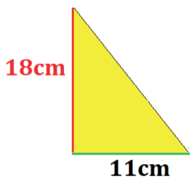


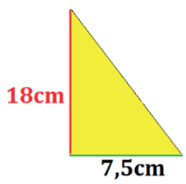

As you can see, the pyramid has two different lateral sides, so it has also two different apothems.
To calculate their lengths we can use the Pythagoras' theorem:




Let's do the calculations:



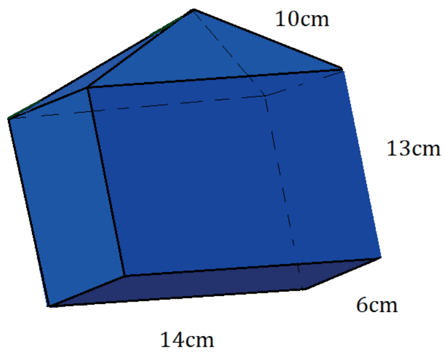

$$\text{apothem}_1 = \sqrt{18^2 + 11^2} \approx 21{,}1 \text{ cm}^2$$
$$\text{apothem}_2 = \sqrt{18^2 + 7{,}5^2} \approx 19{,}5 \text{ cm}^2$$

Therefore, the area of the pyramid is:

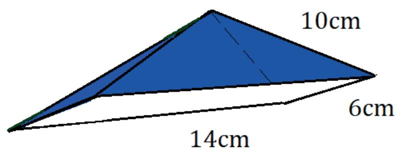
$$A = A_{lateral} + A_{base} \approx 2 \cdot \frac{22 \cdot 21{,}1}{2} + 2 \cdot \frac{15 \cdot 19{,}5}{2} + 15 \cdot 22 = 1086{,}7 \text{ cm}^2$$

★ Sometimes, we need to calculate the area of a **composite 3D-shape**. In those cases, we can try to divide it into several polyhedrons and the area can be obtained by adding their corresponding areas.

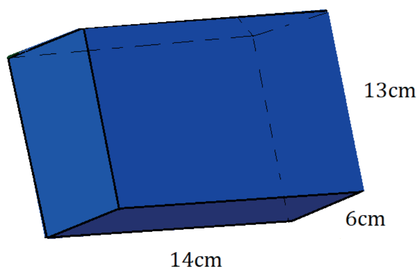
Example: " Calculate the area of the following 3D-shape:"




We can divide this shape into a rectangular prism and a rectangular pyramid.




$$A_{pyramid} = 2 \cdot \frac{14 \cdot \sqrt{10^2 - 7^2}}{2} + 2 \cdot \frac{6 \cdot \sqrt{10^2 - 3^2}}{2} =$$
$$\approx 99{,}98 + 57{,}24 = 157{,}22 \text{ cm}^2$$

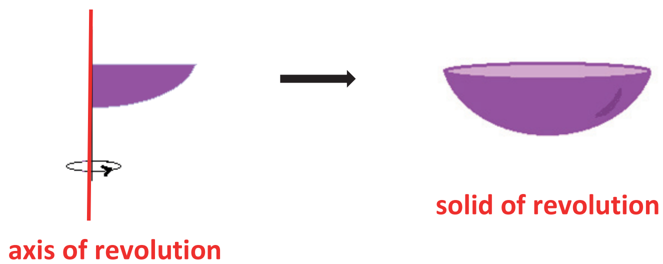



$$A_{prism} = 14 \cdot 6 + 2 \cdot 6 \cdot 13 + 2 \cdot 14 \cdot 13 = 604 \text{ cm}^2$$

Therefore, the total area of the 3D-shape is: $A_{3D\text{-}shape} \approx 157{,}22 + 604 = 761{,}22 \text{cm}^2$

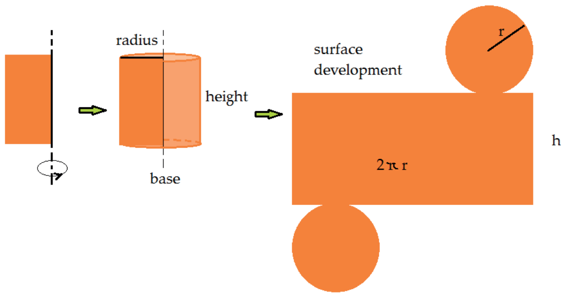
## 5. Solids of revolution. Area of a solid of revolution

Remember that a solid of revolution is a solid figure obtained by rotating a plane shape around a straight line called axis of revolution. Obviously, there are infinite solids of revolution depending on the plane shape we take to generate it.



Let's take some of them and remember the formulas we can use to calculate their corresponding areas:

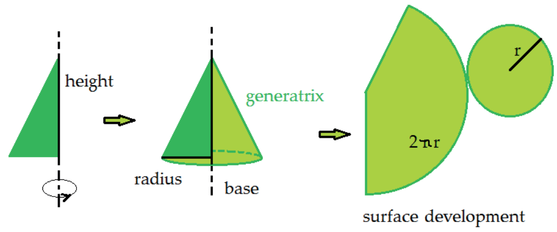
## CYLINDER




*The* ***area*** *of a cylinder is:*

$A = A_{lateral} + 2 \cdot A_{base} = 2\pi r \cdot h + 2 \cdot \pi r^2$

## CONE



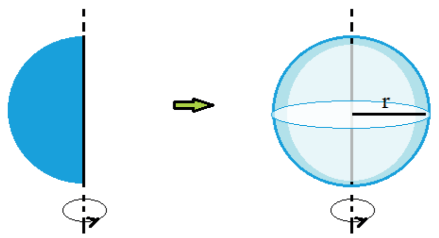

*The* ***area*** *of a cone is:*

$A = A_{lateral} + A_{base} = \frac{2\pi r \cdot \text{generatrix}}{2} + \pi r^2 =$

$A = \pi rg + \pi r^2$

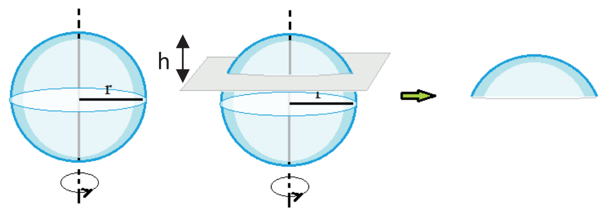
Circular sector Circle

## SPHERE




*The* ***area*** *of a sphere is:* $A = 4\pi r^2$

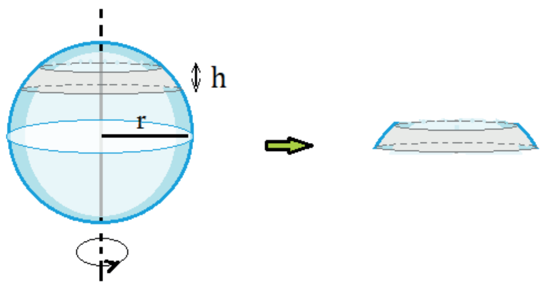
## SPHERICAL CAP




*The* ***area*** *of a spherical cap is:* $A = 2\pi rh$

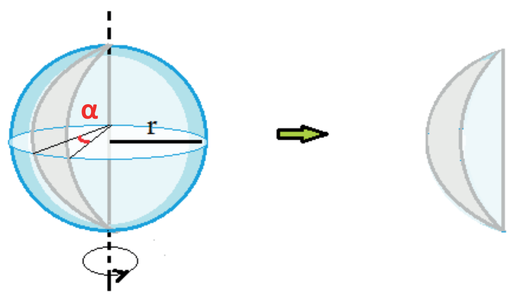
There are also some other **spherical shapes** whose areas are recommendable to remember:

## SPHERICAL SEGMENT




*The* ***area*** *of a spherical segment is:* $A = 2\pi rh$

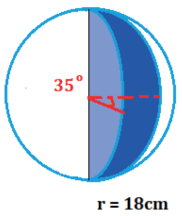
## SPHERICAL WEDGE ( UNGULA)




*The* ***area*** *of a spherical wedge is:* $A = \frac{4\pi r^2 \alpha}{360}$

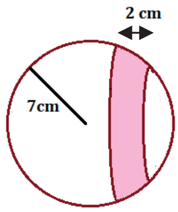
Examples: "Calculate the areas of the following shapes:"

a)



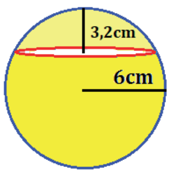

$$A = \frac{4\pi \cdot 18^2 \cdot 35}{360} \approx 395,84 \text{ cm}^2$$

b)




$$A = 2 \cdot \pi \cdot 7 \cdot 2 \approx 87,96 \text{ cm}^2$$

c)

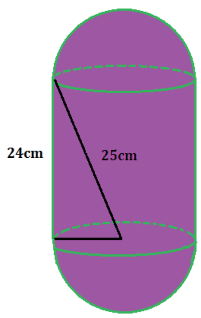



$$A = 2 \cdot \pi \cdot 6 \cdot 3,2 \approx 120,64 \text{ cm}^2$$

As you already know, sometimes we need to calculate the area of **composite 3D-shapes that include shapes of revolution.** In those cases, we can try to divide the shape into several shapes and the area can be obtained by adding their corresponding areas.

Examples: "Calculate the areas of the following composite shapes:"

a)

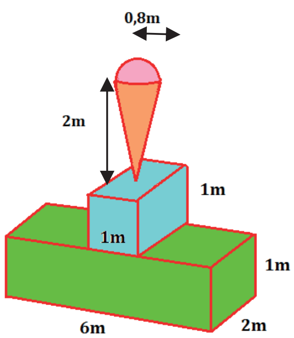



If you have a closer look at the drawing, it is formed by two halves of a sphere and a cylinder without bases. Then, we can add the two halves and calculate the area in an easy way:

$$A = A_{\text{cylinder without bases}} + A_{\text{sphere}} = 2\pi \cdot \left(\sqrt{25^2 - 24^2}\right) \cdot 24 + 4\pi \cdot \left(\sqrt{25^2 - 24^2}\right)^2 \approx$$

$$1671,33 \text{ cm}^2$$

b)

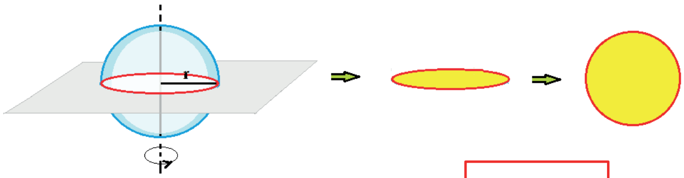



$$A = A_{\text{cone without base}} + A_{\text{semi-sphere}} + A_{\text{prism 1}} + A_{\text{prism 2}} =$$

$$= \pi \cdot 0,4 \cdot \sqrt{2^2 + 0,4^2} + \frac{4\pi \cdot 0,4^2}{2} + (3 \cdot 1 \cdot 2 + 2 \cdot 1^2) +$$

$$+ (2 \cdot 6 \cdot 1 + 6 \cdot 2 - 2 \cdot 1 + 2 \cdot 2 \cdot 1) \approx 29,57 \text{ m}^2$$

Remember also that a circle is the intersection of a sphere:



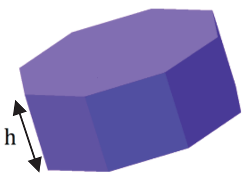

and a plane and its area is: $\boxed{A = \pi r^2}$

## 6. Volume of a 3D - shape

The volume of a 3D-shape is a magnitude that expresses the space it takes up. It is also defined as the quantity of water the 3D-shapes displaces when it is submerged. The volume is measured in cubic units.

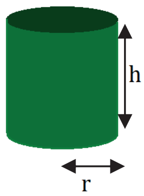
Let's remember the formulas of some 3D-shapes:

**PRISM**



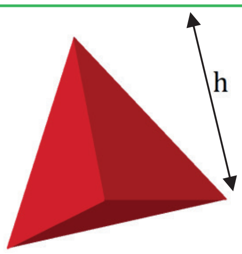

$$V = A_{base} \cdot h$$

**CYLINDER**



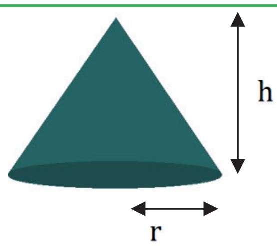

$$V = A_{base} \cdot h = \pi r^2 \cdot h$$

**PYRAMID**



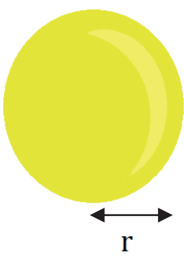

$$V = \frac{1}{3} \cdot A_{base} \cdot h$$

**CONE**




$$V = \frac{1}{3} \cdot \pi r^2 \cdot h$$

**SPHERE**




$$V = \frac{4}{3} \cdot \pi r^3$$

**SOME USEFUL CURIOSITIES**

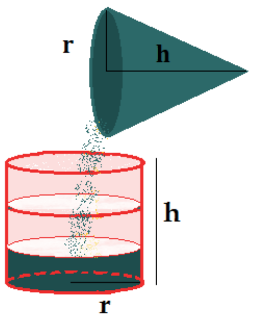
There are some curiosities that may help you remember the previous formulae:

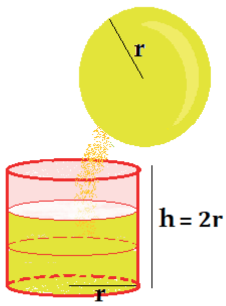
$$V_{cone} = \frac{1}{3} V_{cylinder}$$

$$V_{sphere} = \frac{2}{3} \cdot V_{cylinder}$$

(The height of the cylinder is double the radius of the sphere)

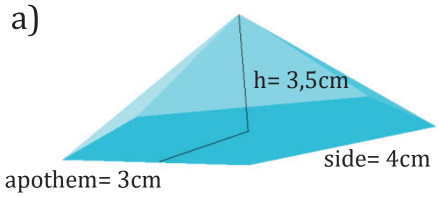
Let's take a look at the following shapes that illustrate the curiosities we have talked about:




Examples: "Calculate the volumes of the following objects:"

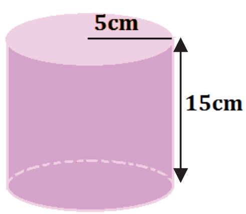
a)




This is a regular pentagonal pyramid, so its volume is:

$$\mathbf{V} = \frac{1}{3}\mathbf{A_{base}} \cdot \mathbf{h} = \frac{1}{3}\,\frac{\text{perimeter} \cdot \text{apothem}}{2} \cdot \mathbf{h}\ \frac{1}{3} \cdot \frac{5\cdot 4\cdot 3}{2} \cdot 3{,}5 = 35\ cm^3$$

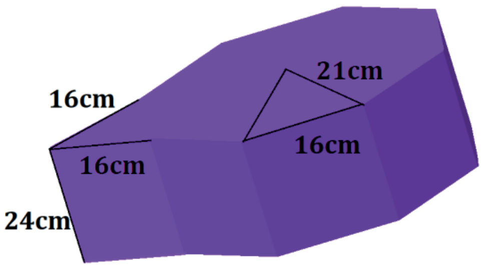
b)




This is a cylinder, then its volume is:

$$\mathbf{V} = \mathbf{A_{base}} \cdot \mathbf{h} = \boldsymbol{\pi} \mathbf{r^2} \cdot \mathbf{h} = \pi \cdot 5^2 \cdot 15 = 1178{,}1\ cm^3$$

c)



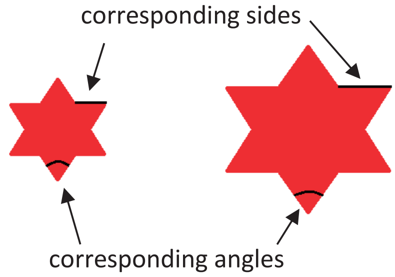

This is a composite shape. We can see it as an octagonal prism joined to a regular triangular prism. Therefore its volume is:

$$\mathbf{V} = \mathbf{V_{triangular\ prism}} + \mathbf{V_{octagonal\ prism}} = \frac{16\cdot\sqrt{16^2-8^2}}{2}\cdot 24 + 8\cdot\frac{16\cdot\sqrt{21^2-8^2}}{2}\cdot 24 = 32484{,}16\ cm^3$$

## 7. Similar polygons. Ratio of similarity

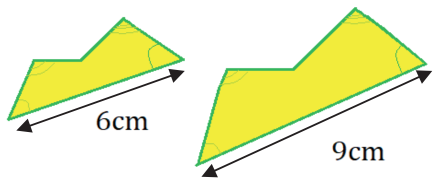
Given two polygons, their corresponding angles as well as their corresponding sides are the ones that appear in the same sequence in the polygons respectively.

Example:




Similar polygons are polygons whose corresponding angles are equal and the lengths of their corresponding sides are proportional. The ratio of similarity is the quotient of one side of one of the polygons and its corresponding side in the other polygon.

Example 1: “Study if the following polygons are similar and if they are similar, calculate the ratio of similarity:”




As you can observe, their corresponding angles are similar and the ratio of similarity is:

$$\frac{9}{6} = 1{,}5$$

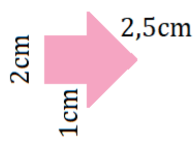
Example 2: “Given the following shape: 

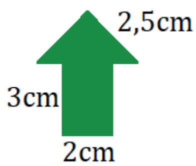
Which of the following shapes are similar to the given one?”

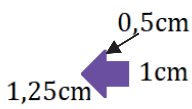
a)  b)  c)  d) 

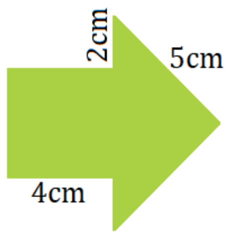
Obviously, a), and d) are similar to the initial one.

Example 3: “Given the following shapes, analyse which of them are similar and the ratio of similarity:

a) 


b) 


c) 


d) 

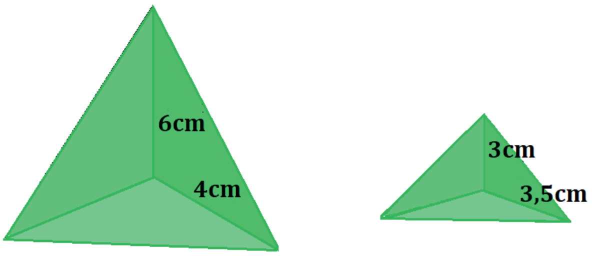
As you can see, a) and b) are not similar because if you check the quotients of their measurements, they are not equal: $\frac{2}{2}=\frac{2,5}{2,5}\neq\frac{3}{1}$

Nevertheless, a) and c) are similar because: $\frac{1}{2}=\frac{1,25}{2,5}=\frac{0,5}{1}$

It is also evident that a) and d) are similar because: $\frac{4}{2}=\frac{5}{2,5}=\frac{2}{1}$

Therefore, a), c) and d) are similar shapes.

Example 4: "Are the following pyramids similar?:"

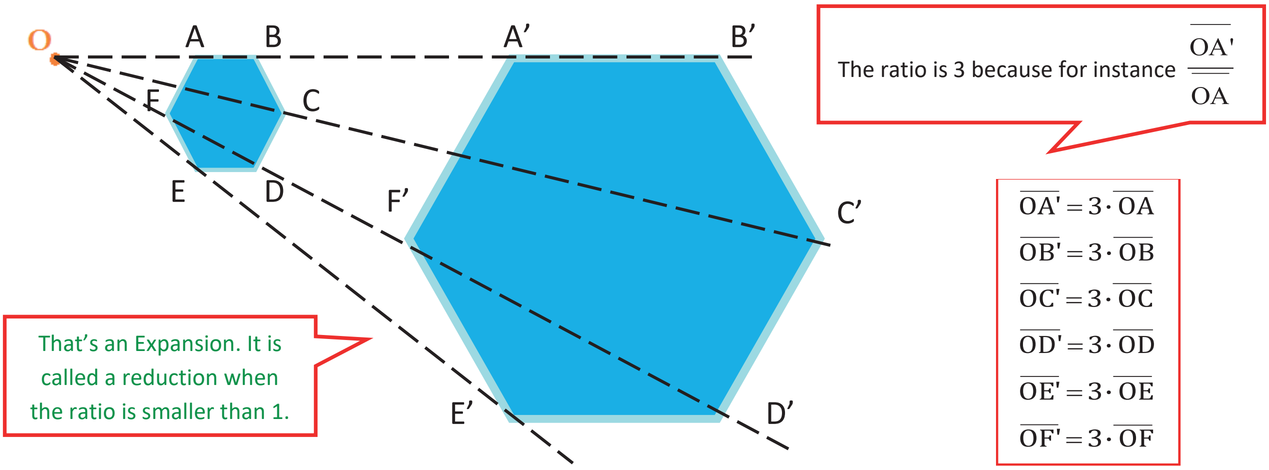



Those pyramids are not similar because:

$$\frac{3}{6}\neq\frac{3,5}{4}$$

Let's remember **how to construct similar polygons**:
First of all, given a polygon, we choose a point of reference ( O ) and draw lines going through that point O and every vertex of the polygon. Then, we draw every transformed point (identified with commas) at the ratio times distance from the point O.

Example: "Observe the following expansion with ratio 3:"

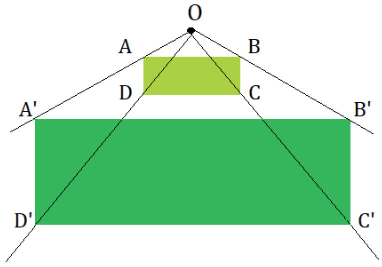



$\overline{OA'}=3\cdot\overline{OA}$
$\overline{OB'}=3\cdot\overline{OB}$
$\overline{OC'}=3\cdot\overline{OC}$
$\overline{OD'}=3\cdot\overline{OD}$
$\overline{OE'}=3\cdot\overline{OE}$
$\overline{OF'}=3\cdot\overline{OF}$

## 8. Ratio of proportionality of areas of similar shapes

Given two similar polygons whose ratio of similarity is r, the ratio of their areas is $r^2$ which is the square of the ratio of their sides.

Example: "Given the following similar rectangles whose ratio of similarity is r = 3 and knowing that the measurements of the smallest one are 4cm and 12cm, calculate the measurements of the bigger one and the ratio of proportionality of their areas."




As the ratio of their sides is 3, we can assure that the measurements of the biggest one are:

$$\frac{\overline{A'D'}}{\overline{AD}} = 3 \longrightarrow \frac{\overline{A'D'}}{4} = 3 \longrightarrow \overline{A'D'} = 12 \text{ cm}$$

$$\frac{\overline{A'B'}}{\overline{AB}} = 3 \longrightarrow \frac{\overline{A'B'}}{12} = 3 \longrightarrow \overline{A'B'} = 36$$

Consequently, the area of each rectangles is:

$A = 4 \cdot 12 = 48 \text{ cm}^2$

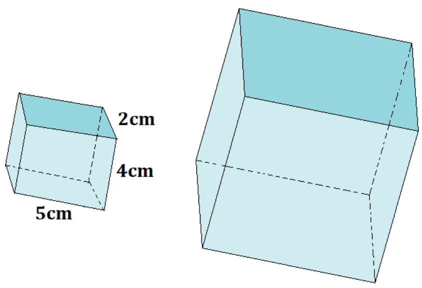
$A = 12 \cdot 36 = 432 \text{ cm}^2$

Therefore, if we calculate the ratio of proportion of their areas, we obtain: $\frac{432}{48} = 9$ which is $3^2$ (**the square** of the ratio of similarity of the rectangles).

## 9. Ratio of proportionality of volumes of similar shapes

Given two similar polyhedrons whose ratio of similarity is r, the ratio of their volumes is $r^3$ which is the cube of the ratio of their sides.

Example: "Given the following similar prisms whose ratio of similarity is r = 2, calculate the measurements of the bigger one and the ratio of proportionality of their volumes."




As the ratio of similarity is 2, the measurements of the bigger one are 4cm, 8cm and 10cm respectively.

Let's calculate their volumes:

$V = 2 \cdot 4 \cdot 5 = 40 \text{ cm}^3$

$V = 4 \cdot 8 \cdot 10 = 320 \text{ cm}^2$

Therefore, if we calculate the ratio of proportion of their volumes, we obtain:

$\frac{320}{40} = 8$ which is $2^3$ (**the cube** of the ratio of similarity of the prisms)

## 10. Similarity in real life

Similarity may appear in many aspects of real life. For instance, similar shapes have numerous applications related to architecture, cartography, construction, design, art, engineering, science…

One concept intrinsically related to similarity is the concept of scale.

Scale is the ratio between a length in a drawing and a length in the real life. Therefore, when we use a scale we are drawing a similar shape to a real world shape.

To express a scale we normally use the following structure:

number of units in the drawing → **a : b** ← number of units in the real life

Example: " The Atomium is a landmark building constructed in Brussels in the past century. It is a big stainless steel structure that represents the shape of a unit animal cell magnified approximately 165 billion times. There are 9 spheres whose diameter is 18m, eight of them are placed in the vertexes of a cube and they are connected by tubes to another central sphere.

a) What is the scale used to construct the building?
165 000 000 000 000 : 1

b) What would be the diameter of a sphere in a cell?

It would be: $\frac{18}{165000000000000} \approx 1{,}09 \cdot 10^{-13}$ m.

As $10^{-12}$ are pico-units in the ISU( International system of units), we can say that the diameter in the real life is approximately 0,109 picometers.



c) Calculate the volume of one of the spheres and what would that volume be in the real life ( in the cell) by using the ratio of proportionality between volumes.

$$V_{\text{sphere of the building}} = \frac{4 \cdot \pi \cdot 9^3}{3} \approx 3053{,}63 \text{ m}^3$$

Therefore, to calculate the volume in the cell we should divide that result by the cube of the ratio of proportionality. Knowing the ratio of proportionality is $165 \cdot 10^{12}$ :

$$V_{\text{sphere of the cell}} = 6{,}8\ 10^{-40}\ m^3$$

d) Calculate the volume of one of the spheres and what would that volume be in the real life ( in the cell) by using the formula of the volume of a sphere.

$$V_{\text{sphere of the cell}} = \frac{4 \cdot \pi \cdot (1{,}09 \cdot 10 - 13 : 2)^3}{3} \approx 6{,}8 \cdot 10^{-40}\ m^3$$

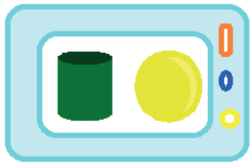
As we can see, the results of sections c) and d) are logically the same.

## COMPLETE THE FOLLOWING TABLE :

| Key VOCABULARY | *phonetics* | meaning |
|---|---|---|
| Polygon | | |
| Regular | | |
| Irregular | | |
| Perimeter | | |
| Area | | |
| Surface | | |
| Quadrilateral | | |
| Composite | | |
| Apothem | | |
| Circular | | |
| Sector | | |
| Segment | | |
| Annulus | | |
| Polyhedron | | |
| Platonic solid | | |
| Prism | | |
| Cuboid | | |
| Convex | | |
| Concave | | |
| Pyramid | | |
| Apex | | |
| Surface development | | |
| Tetrahedron | | |
| Solid of revolution | | |
| Axis of revolution | | |
| Cylinder | | |
| Cone | | |
| Sphere | | |
| Spherical cap | | |
| Spherical wedge | | |
| Ungula | | |
| Similar | | |
| Ratio of similarity | | |
| Ratio of proportionality | | |

# UNIT 5

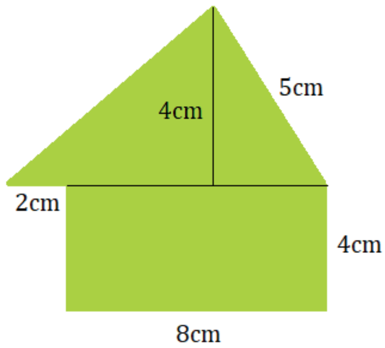
## *EXERCISES*



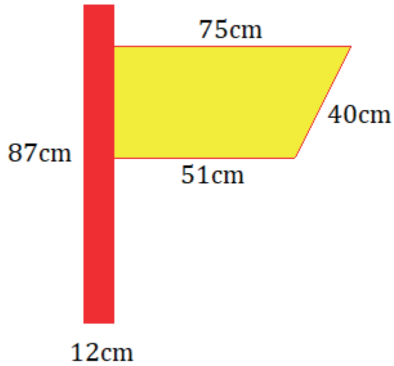
## PERIMETERS AND AREAS OF 2-D SHAPES

1. Calculate the perimeter and the area of the following composite shapes:

a)




b)




2. Calculate the area and the perimeter of a shape formed by six identical rhomboids joined by one common vertex if the base, the side and the height of each rhomboid are 10cm, 15cm and 4cm respectively.

3. Calculate the perimeter and the area of a rectangle whose diagonal is 65cm long and one of its sides is 13cm long.

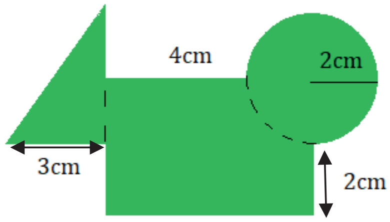
4. Calculate the perimeter and the area of an isosceles trapezoid whose bases are 9cm and 15cm and whose equal sides are 5cm each.

5. Calculate the perimeter and the area of a rhombus whose side is 85cm and whose short diagonal is 26cm.

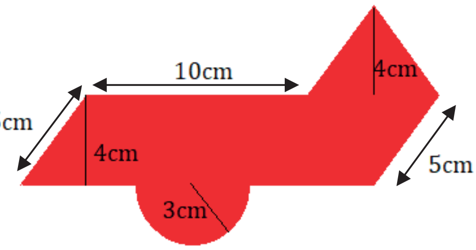
6. Calculate the perimeter and the area of a square whose diagonal measures $5\sqrt{2}$ cm.

7. Given the following shapes, calculate their perimeters and areas:

a)

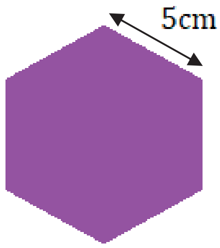



b)

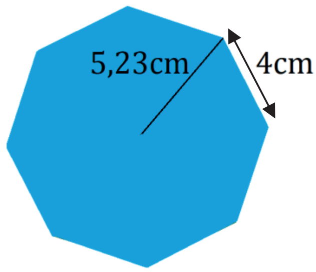



8. Calculate the perimeters and areas of the following regular polygons:

a)




b)

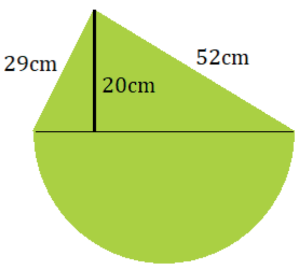



9. Calculate the perimeter and the area of a regular heptagon whose side is 3m and whose apothem is 3,62m.

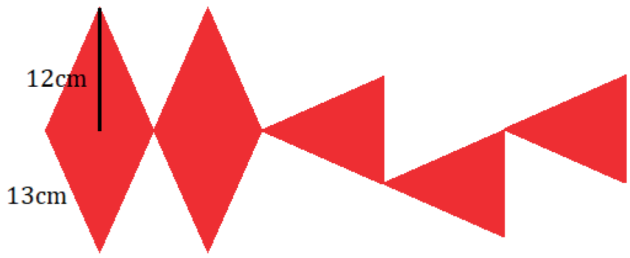
10. Calculate the perimeter and the area of a regular enneagon whose radius is 7cm and whose side is 4,79cm.

11. Calculate the perimeter and the area of the following composite shapes:

a)

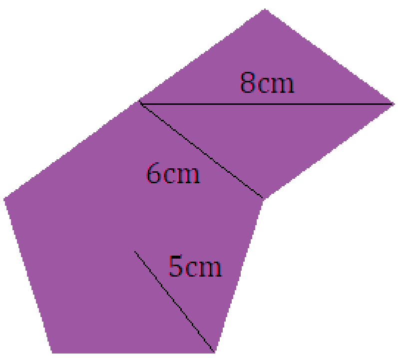



b)

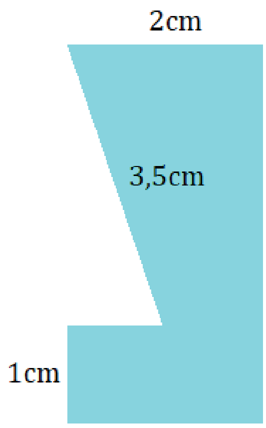



12. Calculate the perimeter and the area of the following composite shapes:

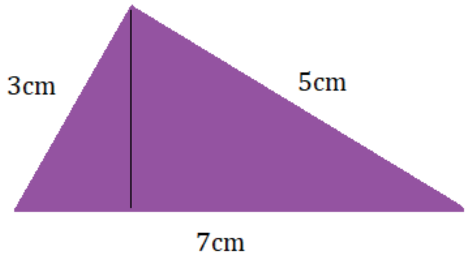
a)




b)

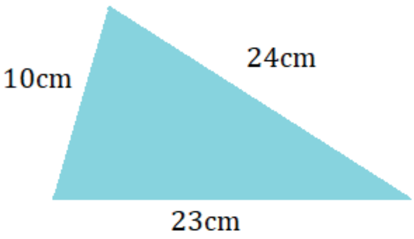


13. Calculate the area of the following triangle and reason if it is a right-angled triangle or not.

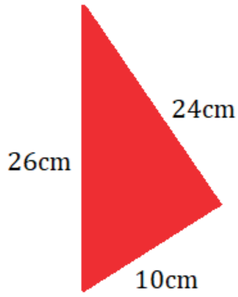



14. Calculate the perimeter and the area of the following triangles and prove if there is any right-angled triangle among them:

a)

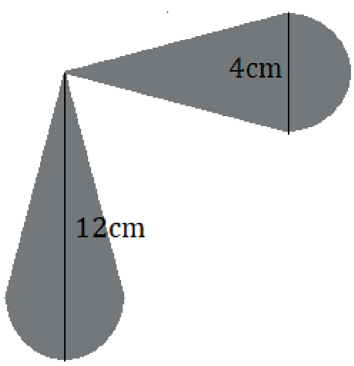



b)

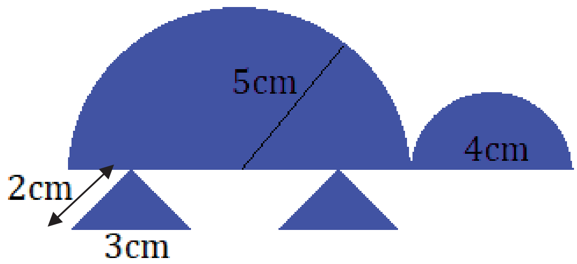



15. Calculate the area of the following shapes:

a)




b)

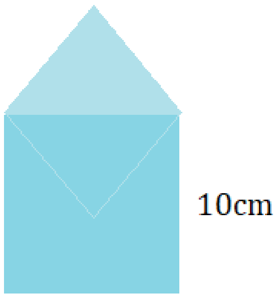



16. Find out if the following measurements correspond to right-angled triangles and then calculate their areas and perimeters:

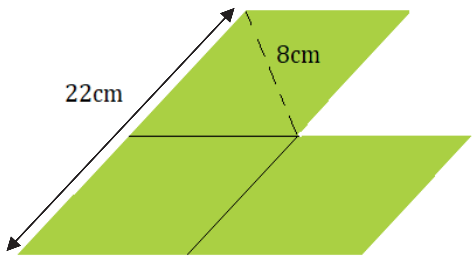
a) 30cm, 16cm and 34cm b) 9m, 40m and 41m c) 22dm, 35dm and 16dm

17. Calculate the perimeters and areas of the following shapes:

a)

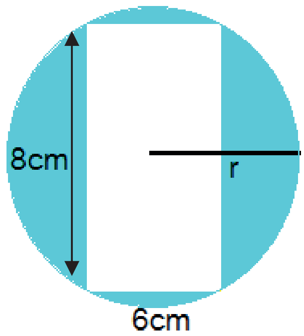



b)

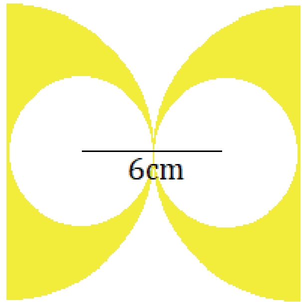


18. Calculate the perimeter and the area of the following shaded shapes:

a)

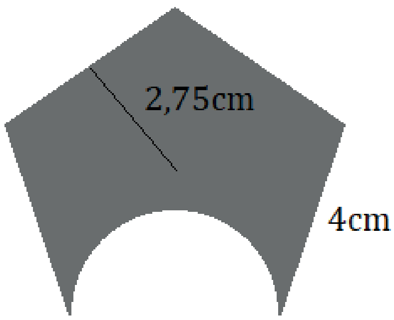



b)



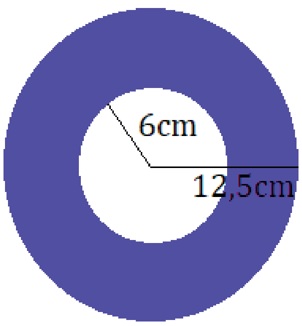

c)


d)

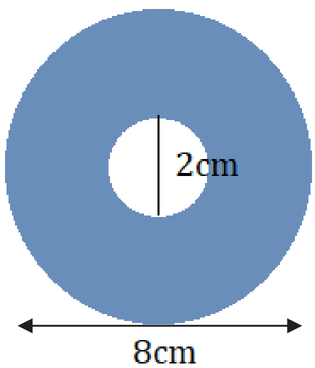



19. Calculate the area and the perimeter of the following circular shapes:

a)

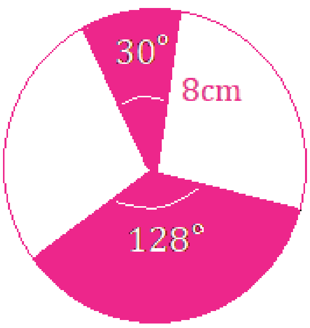



b)

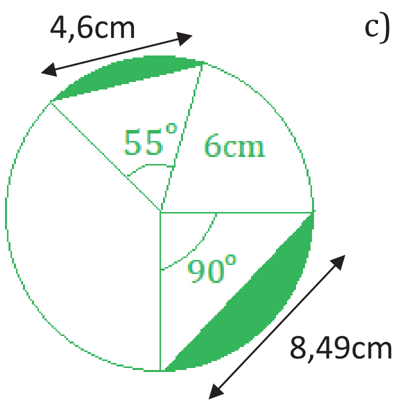



20. Calculate the area and the perimeter of the following shaded shapes:

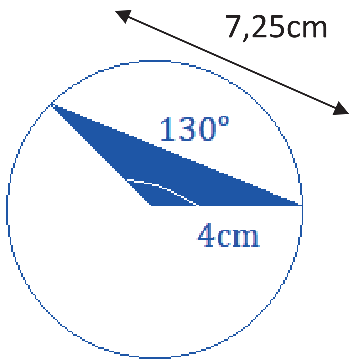
a)




b)

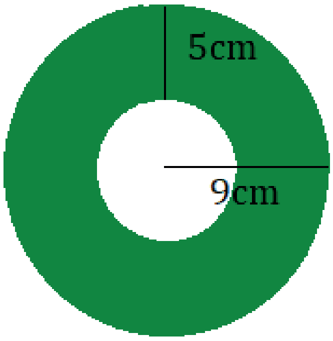



c)

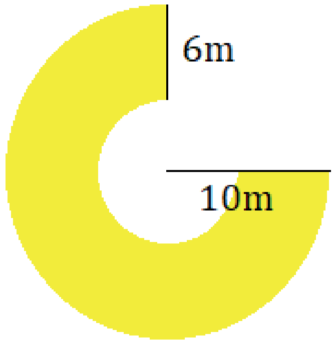


21. Calculate the area of the following shapes:

a)

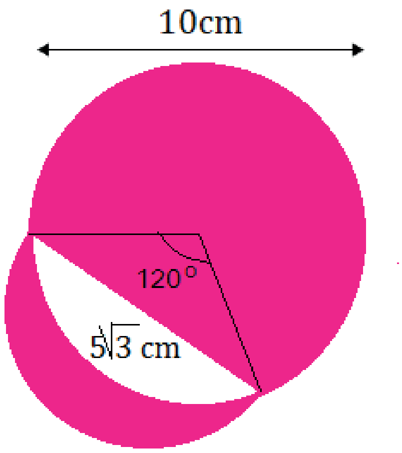



b)

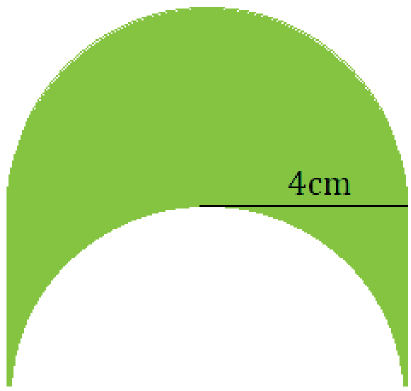



22. Calculate the area of the following shapes:

a)




b)




## AREAS OF POLYHEDRONS AND OTHER 3-D SHAPES

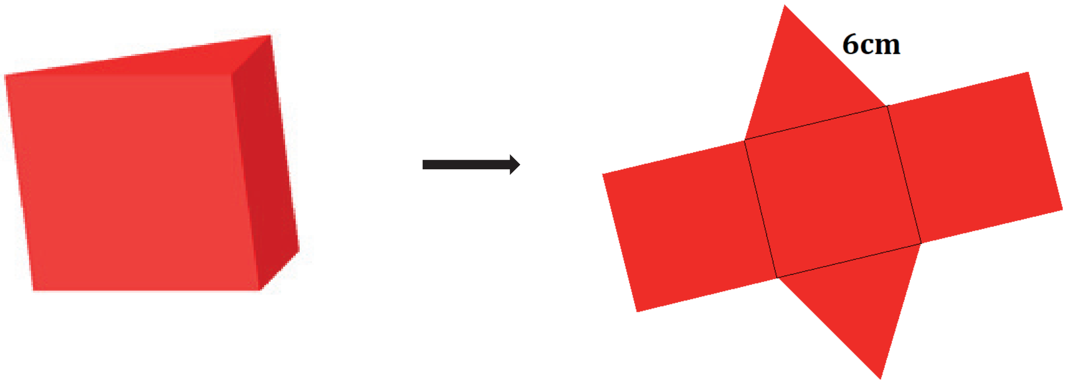
23. Calculate the area of a regular dodecahedron whose edge is 5cm and the apothem of each side is 3,44cm.

24. Calculate the area of a regular icosahedron whose edge is 2cm.

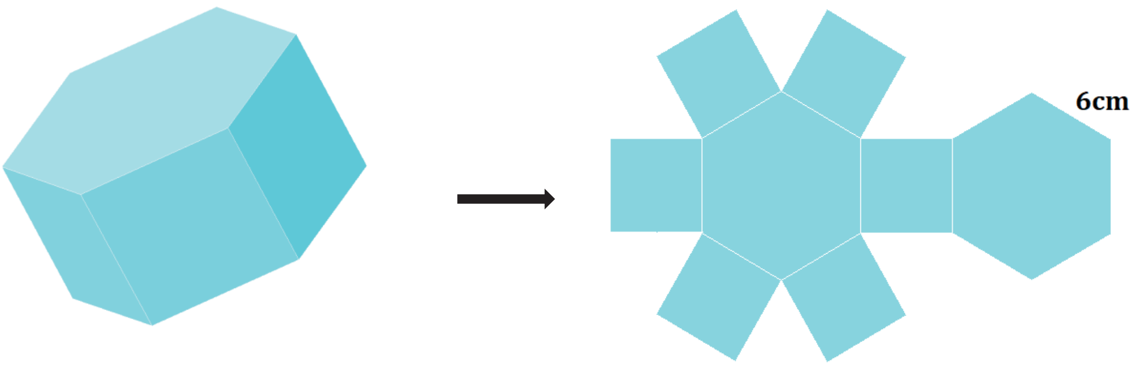
25. Calculate the area of a regular octahedron where the height of one of its sides is $2\sqrt{5}$ cm.

26. Calculate the area of the following **prisms**:

a)




a)




27. Calculate the area of the prisms described:

a) A right quadrangular prism whose height is 12cm and the side of its base is 3cm.

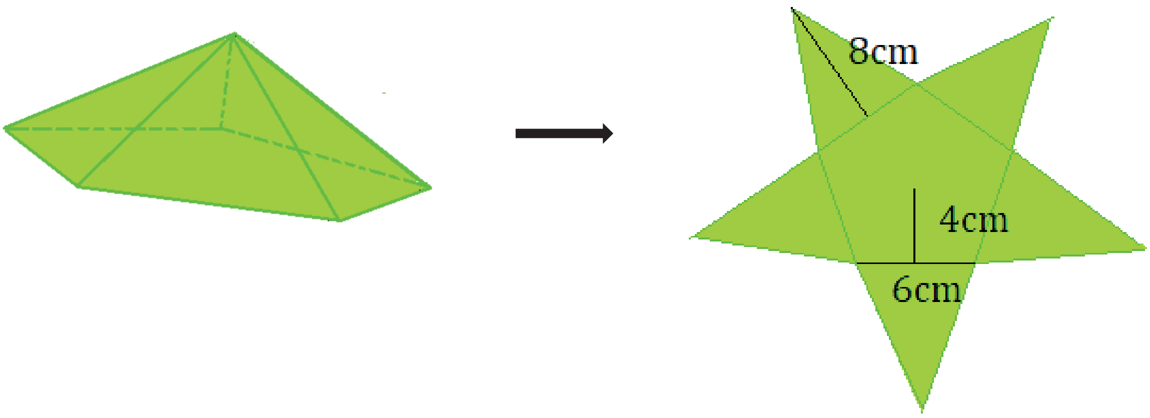
b) A right hexagonal prism whose height is 8cm and the side of its base is 3cm.

28. Given a cube whose interior diagonal is $\sqrt{432}$ cm, calculate its area.

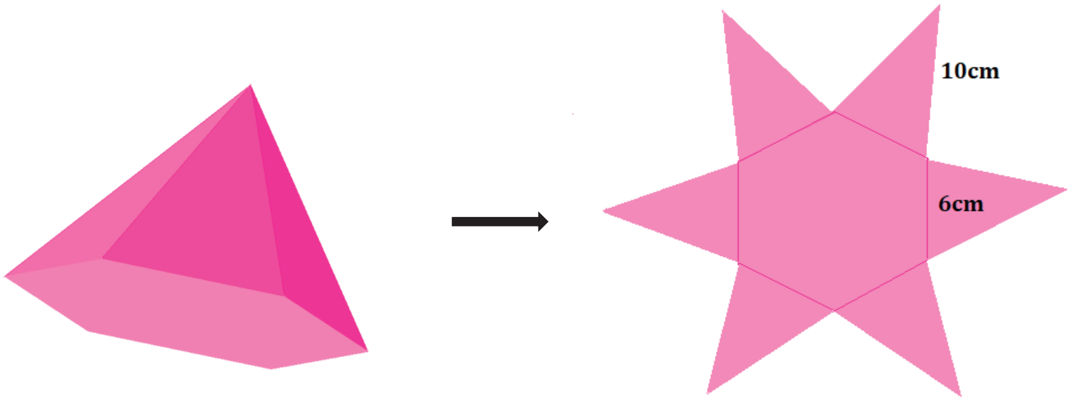
29. Given a cube whose area is 24cm$^2$, calculate its interior diagonal.

30. Calculate the area of the following **pyramids**:

a)



b)




31. Calculate the area of the pyramids described:

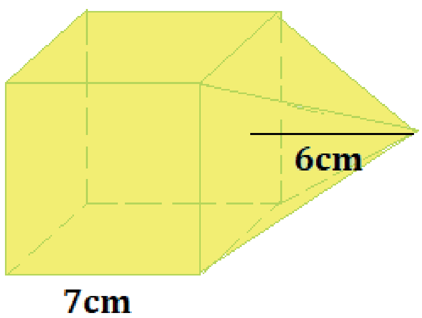
a) A right quadrangular pyramid whose height is 10cm and the side of its base is 3cm.

b) A right octagonal pyramid whose height is 12m and the side and apothem of its base are 2,4m and 2m respectively.

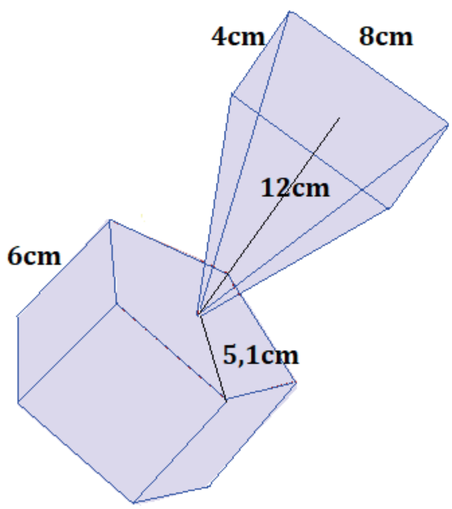
c) A right hexagonal pyramid whose height is 8cm and the side of is base is 1cm.

32. Calculate the area of the following **composite 3-D shapes**:

a)



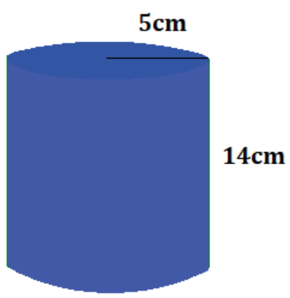

b)



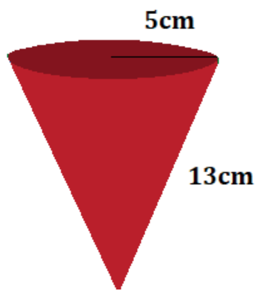

## AREAS OF SOLIDS OF REVOLUTION and SPHERICAL SHAPES

33. Calculate the areas of the following **solids of revolution**:

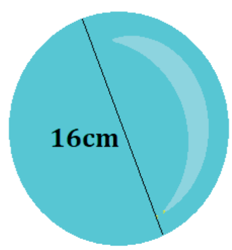
a)




b)




c)



34. Calculate the area of the shapes described:

a) A cylinder whose height is 15cm and whose radius is 2cm.

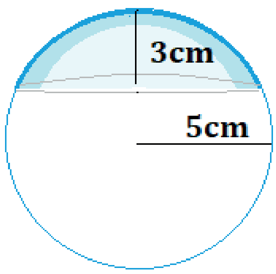
b) A sphere whose diameter is 3cm.

c) A cone whose generatrix is 10cm and the radius of its base is 2cm.

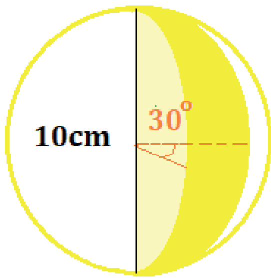
d) A cylinder whose height is 20dm and whose diameter is 3dm.

35. Calculate the areas of the following **spherical shapes**:

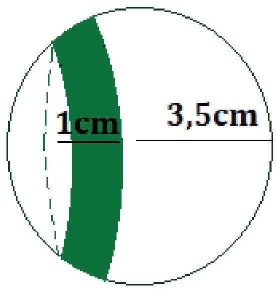
a)




b)

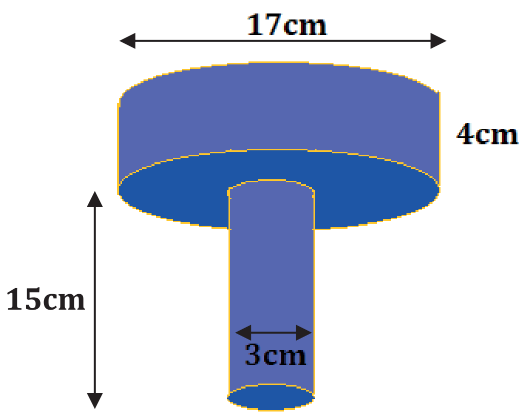



c)

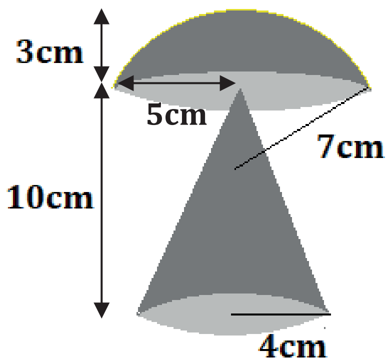



36. Calculate the areas of the following composite shapes:

a)

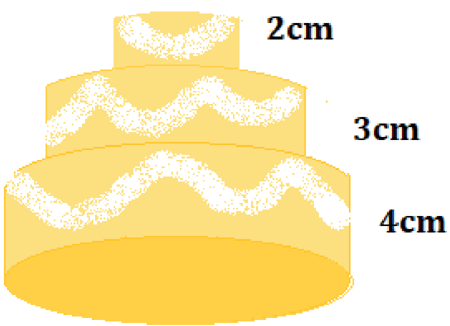



b)

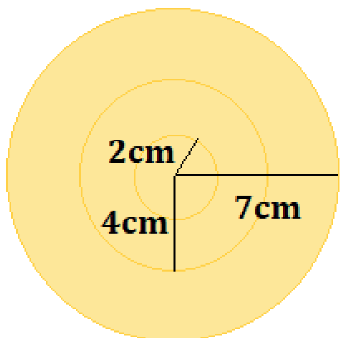



37. Calculate the total areas of the following composite shapes:

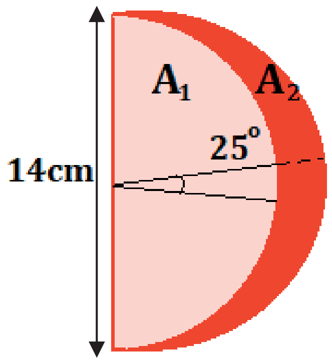
a)




**viewed from above**



b)



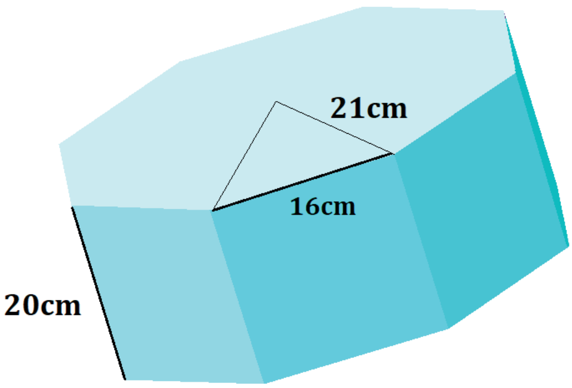

This shape is formed by:

$A_1$ , $A_2$ , and $A_3$ ( what is at the back and it is equal to $A_1$)

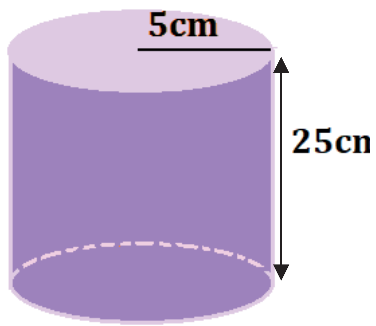
## VOLUMES OF 3-D SHAPES

38. Calculate the volumes of the following shapes:

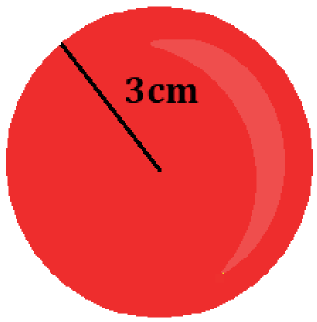
a)



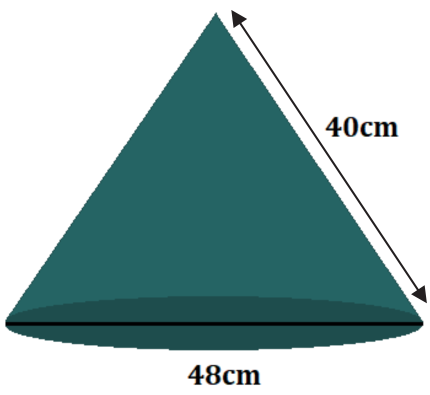

b)




c)




d)



39. Calculate the volumes of the above described shapes:
a) A sphere whose diameter is 17cm.
b) A cylinder whose height is 22cm and the radius of one of its bases is 3cm.
c) A right hexagonal prism whose height is 19cm and the side of its base is 2cm.
d) A hexagonal pyramid whose height is 30cm and the side of its base is 1cm.
e) Half a sphere whose radius is 2cm.
f) A pentagonal pyramid whose height is 11cm and the side and radius of its base are 6cm and 5,1cm respectively.

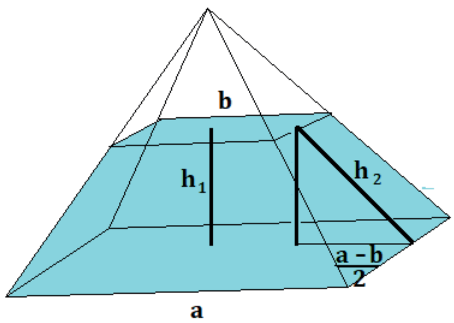
## Areas and volumes of **Truncated shapes**

Remember that a Truncated shape is a shape that is the result of cutting a shape by a plane parallel to a base and taking the part of that base.

**Two tips**:

- Take into account the **Pythagoras' theorem** as it is shown in the case of a truncated pyramid:

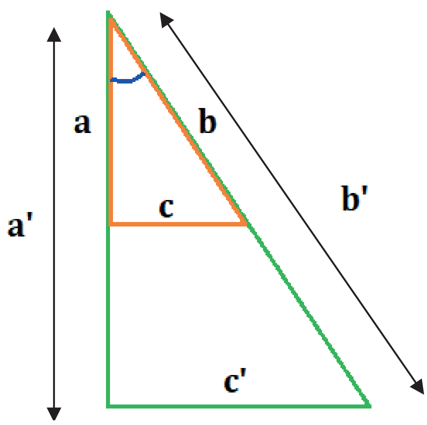
$$h_2 = h_1^2 + \left(\frac{a-b}{2}\right)^2$$



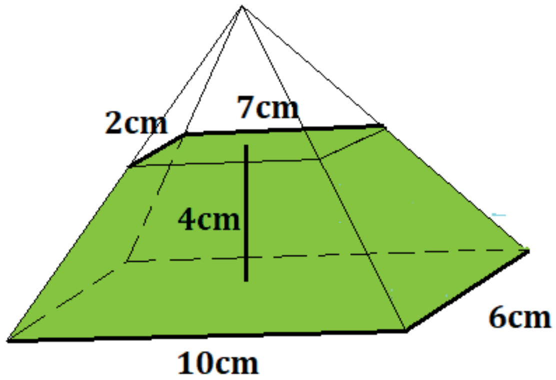

- Take into account the **Thales' theorem** so that we can use the rules of proportionality and similarity of triangles as it is shown in the following drawing:

Given two triangles with a common angle and parallel opposite sides, they are similar triangles and have proportional sides.
Therefore, we can assure that:

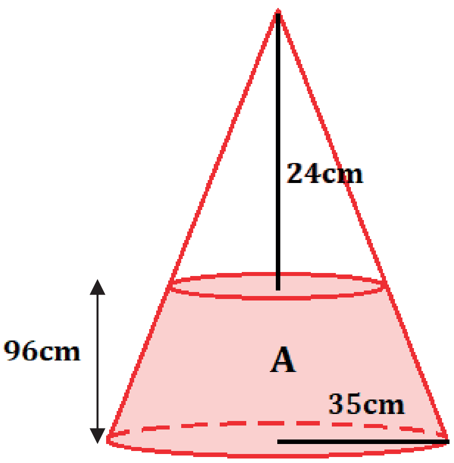
$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$



40. Calculate the area of the following truncated pyramid:

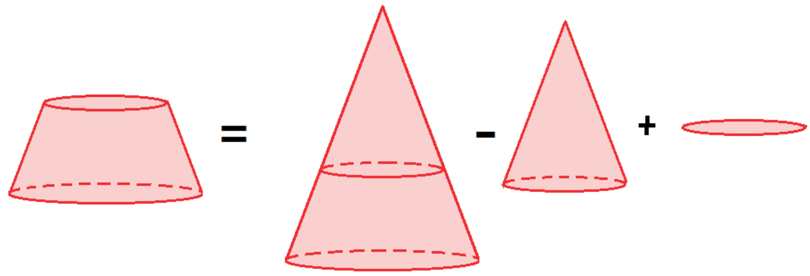



41. Calculate the area of the following truncated cone:



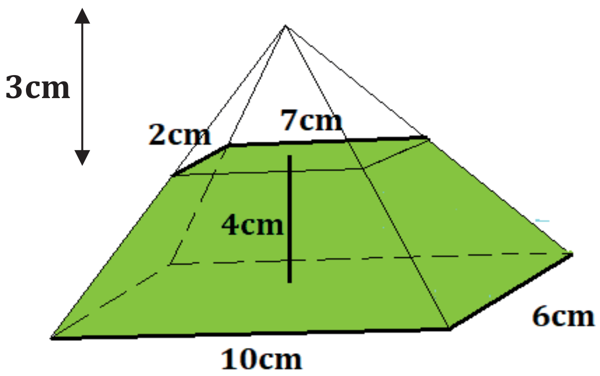

ONE TIP: You can calculate A in this way:

$$A = A_{\text{big cone}} - A_{\text{small cone}} + \text{upper base}$$

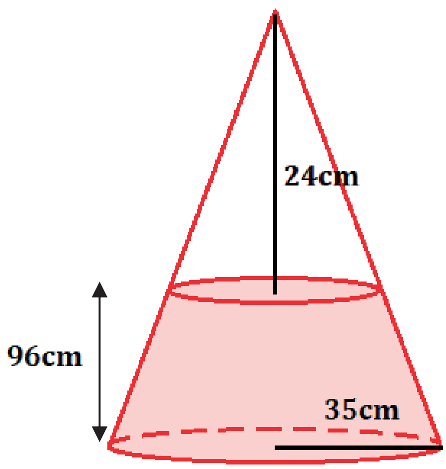


42. Calculate the volume of the following shapes:

a)

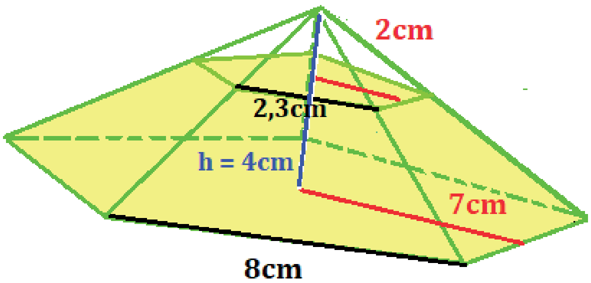



b)

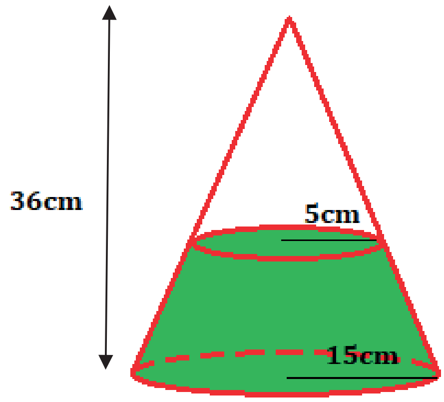


43. Calculate the area and the volume of the following **Truncated shapes**:

a)

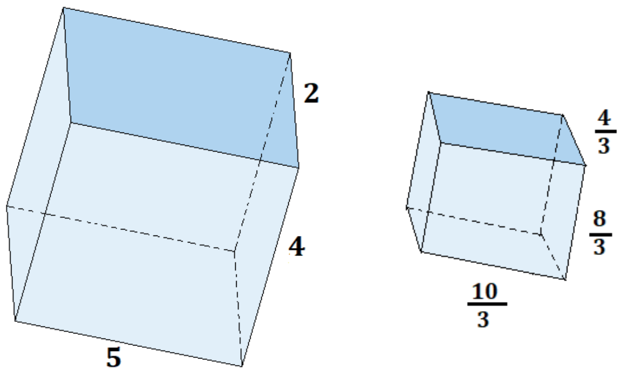



b)



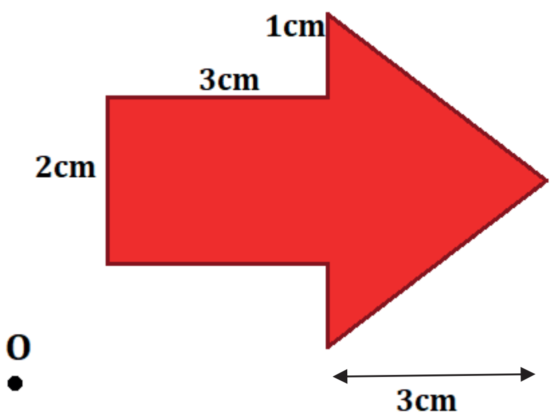

## SIMILARITY

44. Given the following shapes, analyse if they are similar and if they are, calculate the ratio of similarity.

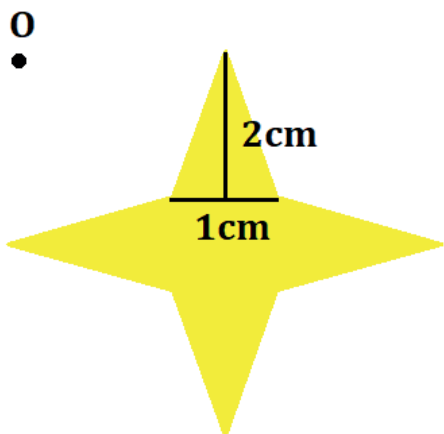



45. Given the following shapes, construct similar shapes with ratio r = 3.

a)




b)



46. Given a rectangle, whose measurements are 14cm and 20cm, calculate the area of the correspondent similar rectangle whose ratio of similarity is r = 1,5.

47. If there are two similar circles whose ratio of similarity is r = 4, calculate its radiuses if you know that the area of the biggest one is $36\pi cm^2$.

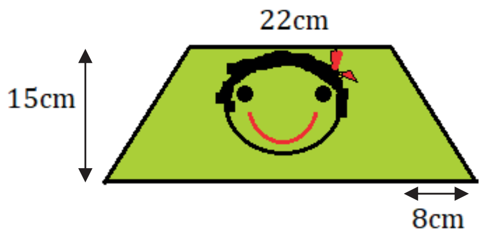
48. Given a rectangular prism whose measurements are 10cm, 12cm and 24cm, calculate the volume of the corresponding similar rectangular prism with ratio of similarity r = 7.

49. Given a cone whose height is 10cm and its radius is 3cm, calculate the volume of the similar cone whose ratio of similarity is r = 8.

50. If the area of a sphere is $100\pi cm^2$, calculate the radius of a similar sphere whose ratio of similarity is r = 7. Calculate the volume of both spheres.

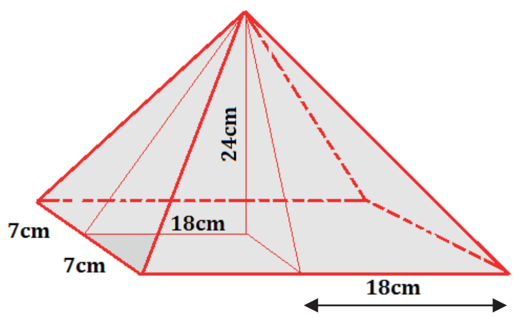
## WORD PROBLEMS

51. A kindergarten teacher wants to decorate her classroom with drawings like the one shown. She wants to cover one of the walls whose surface is $14m^2$. Estimate the minimum number of drawings she needs to draw to cover that wall.




52. Calculate the area of a circle circumscribed to a right-angled triangle whose legs are 5cm and 12cm long and the hypotenuse coincides with the diameter of the circle.

53. A sculptor has created a model of a pyramid whose measures are shown in the drawing. He wants to paint it with grey paint. How many tins of paint does he need if he can paint $6cm^2$ with each tin?



54. A company that makes tins for chips, decides to change the shape of the tins. They use to be cylindrical tins 20cm height and whose radius were 5cm. Now, they are planning to use rectangular prisms whose measurements are 5cm, 7cm and 22cm. What option would be more interesting for the company economically speaking? (they try to reduce the surface to reduce the quantity of materials)

55. In a big square of an important city, there is a statue of a well-known mathematician. It is made of marble and its surface is $6m^2$. The owner of a company wants to do a model of that statue.
a) If she wanted to paint it with silver paint, what quantity of paint would she need to cover the surface of the model made at a **reduction scale** of 30 : 1?
b) How much would the paint for the model be if the silver paint costs 1,5 euros by $cm^2$ ?

56. A table tennis player won an important prize last winter and he wants to have a sculpture made of the racket he used to win the tournament. He asked an artist to make the sculpture who decided to do it at amplification scale of 1 : 50. What would be the measurements of the sculpture if the corresponding measurements of the racket are 5mm, 150mm and 300mm ?