MATHEMATICS

4º ESO

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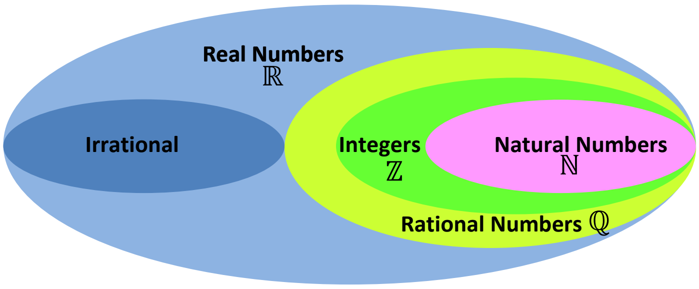
LESSON 1: REAL NUMBERS

Keywords

REAL diagonal RATIONAL **irrational** radicand golden section **divine proportion** INTERVAL SINGLE *phi* *root* Thales' theorem *prove-proof* endpoint radical FRACTION non-ending Set-builder parenthesis indices infinity symbol SCIENTIFIC NOTATION rationalize

1. NUMBERS

So far, you have studied different sets of numbers: natural numbers, integers, fractions or rational numbers and real numbers.



Remember: A ***set*** is a collection of objects, these objects are called ***elements***.

- *Natural (or counting) Numbers* ($\mathbb{N}$):0, 1, 2, 3, 4 …
- *Integers* ($\mathbb{Z}$): …, -3, -2, -1, 0, 1, 2, 3, …
- *Rational numbers* ($\mathbb{Q}$): Natural numbers, integers and fractions are **rational** numbers. A **fraction** is formed by two numbers like:

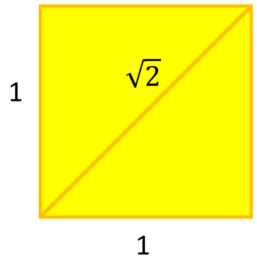
$$\begin{array}{l} \underline{5} \leftarrow \text{Numerator} \\ 3 \leftarrow \text{Denominator} \end{array}$$

- *Irrational numbers*: They are numbers which can't be expressed as fractions. $\sqrt{2}, \sqrt{3}, \pi, \phi \ldots$
- *Real numbers* ($\mathbb{R}$): It's the set formed by rational and irrational numbers.

1.1. Irrational numbers

You know a lot about rational numbers, so we are going to study irrational numbers more deeply in this lesson.

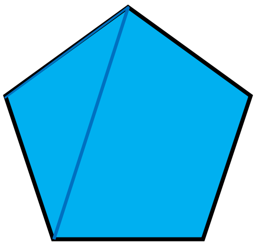
- $\sqrt{2}$ is an irrational number, so it can’t be expressed as a fraction. Let’s prove it!



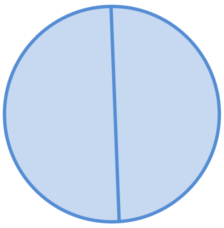
Proof: Let’s suppose that $\sqrt{2}=\frac{p}{q}$ (proof by contradiction), then $2=\frac{p^2}{q^2} \Rightarrow p^2=2q^2$ · p^2 is a perfect square and it’s also an even number, so it contains the factor 2 an even number of times, and then q^2 must be even too and it contains the factor 2 an even number of times, but taking into account that $p^2=2q^2$ we get that p^2 has the factor 2 an odd number of times. Finally there is a contradiction in the underlined sentences because $\sqrt{2}$ can’t be written as a fraction.

- The ***golden*** number $\phi = \frac{1+\sqrt{5}}{2}$ is an irrational number. It’s also known as ***golden section*** and ***divine proportion***.

  The ratio between the diagonal and the side of a regular pentagon is ϕ.



- π is the relation between the length of a circumference and its diameter.



$$L = 2\pi r = \pi d \;\; \Rightarrow \;\; \pi = \frac{L}{d}$$

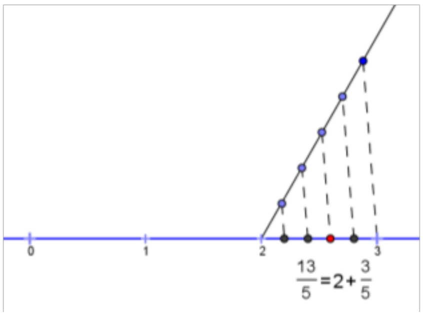
$$d = diameter, \;\; r = radius$$

$$L = length$$

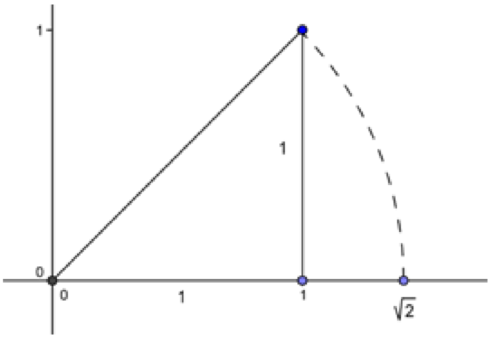
1.2. Number line

We can represent numbers on the real number line.

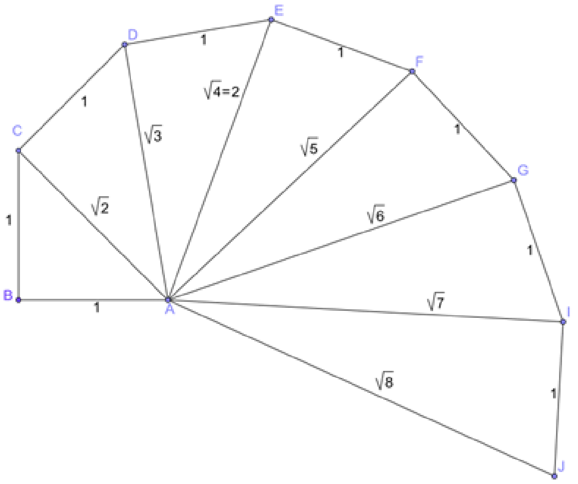
- Rational numbers: you can represent them using Thales’ Theorem.



- Irrational numbers: you can represent some of them using Pythagoras' Theorem. Look at this example:



Exercise: Describe this picture where every triangle is a right triangle.



2. INTERVALS

An interval is a set formed by the real numbers between, and sometimes including, two numbers. They can also be non-ending intervals as we are going to see.

(means "not included" or "open"

[means "included" or "closed"

- Open interval **(1, 3)**, formed by all the real numbers between 1 and 3, but the ***endpoints*** are **not included**.

$$(1,3)=\{x \epsilon \mathbb{R} : 1 < x\ 3\}$$

*Note: The notation between {} is called **set-builder** notation.*

- Closed interval **[1, 3]**, formed by all the numbers between 1 and 3 where the ***endpoints*** are **included**.

$$[1,3]=\{x \epsilon \mathbb{R} : 1 \leq x \leq 3\}$$

- Half-open interval (left-closed, right-open) **[1, 3)** which contains the real numbers between 1 and 3. 1 is included but 3 is not.

$$[1,3)=\{x \epsilon \mathbb{R} : 1 \leq x < 3\}$$

- Half-open interval (left-open, right-closed) **(1, 3]** which contains the real numbers between 1 and 3. 1 is not included but 3 is included.

$$(1,3]=\{x \epsilon \mathbb{R} : 1 < x \leq 3\}$$

Some intervals don't end, they are called ***non-ending*** intervals. One (or both) endpoints are $+\infty$ and $-\infty$. We always use the parenthesis "**(**" with these symbols. There are different possibilities:

- **(1, +∞)=** $\{x \epsilon \mathbb{R} : x > 1\}$ Observe that 1 is not included.
- **(−∞, 3)=** $\{x \epsilon \mathbb{R} : x < 3\}$ Observe that 3 is not included.
- **[1, +∞)=** $\{x \epsilon \mathbb{R} : x \geq 1\}$ Observe that 1 is included.
- **(−∞,** 3**]=** $\{x \epsilon \mathbb{R} : x \leq 3\}$ Observe that 3 is included.
- **(−∞, +∞)=** $\mathbb{R}$

Exercise: Express these intervals using set-builder notation.

a) [-4, 3]={ }

b) (2, 6]= { }

c) (−∞, 4)= { }

d) [0, +∞)= { }

3. RADICALS

Remember:

$$\sqrt[n]{a} = b \Rightarrow b^n = a$$

$$\sqrt[n]{a^m} = a^{\frac{m}{n}}$$

Properties of radicals

1. The n-th root of a negative number doesn't exist if n is an even number.

   Example: $\sqrt[4]{-81}$ doesn't exit.

2. The n-th root, where n is an even number, of a positive number has two different solutions, one is positive and the other one is negative.

   Example: $\sqrt{16} = \pm 4$.

3. You can simplify radicals expressing them as powers.

   Example: $\sqrt[4]{25} = \sqrt[4]{5^2} = 5^{\frac{2}{4}} = 5^{\frac{1}{2}}$.

4. You can convert radicals with different indices into radicals with the same index. First, express the radicals as powers. Secondly, convert the fractions into fractions with the same denominator.

   Example: $\sqrt{7}$ and $\sqrt[3]{10} \Longrightarrow 7^{\frac{1}{2}}$ and $10^{\frac{1}{3}} \Longrightarrow 7^{\frac{3}{6}}$ and $10^{\frac{2}{6}} \Longrightarrow \sqrt[6]{7^3}$ and $\sqrt[6]{10^2}$

5. You can multiply radicals with the same index.

   Example: $\sqrt[3]{5} \cdot \sqrt[3]{7} = \sqrt[3]{5 \cdot 7} = \sqrt[3]{35}$

6. You can divide radicals with the same index.

   Example: $\frac{\sqrt[4]{45}}{\sqrt[4]{9}} = \sqrt[4]{\frac{45}{9}} = \sqrt[4]{5}$

7. You can multiply and divide radicals with different indices. First, you need to convert them into radicals with the same index.

   Example: $\sqrt[3]{5} \cdot \sqrt[4]{2} = \sqrt[12]{5^4} \cdot \sqrt[12]{2^3} = \sqrt[12]{5^4 \cdot 2^3}$

8. To calculate the power of a root you raise the radicand to the power.

   Example: $\left(\sqrt[3]{2}\right)^5 = \sqrt[3]{2^5}$

9. You can simplify roots factoring the radicand and "taking out" the factors whose exponents are equal or greater than the index. Divide the exponent by the index and the quotient is the exponent of the factor you can "take out", the remainder is the exponent of the factor inside the radical. It will be easier to understand with this example.

   Example: $\sqrt[3]{2^5 \cdot 3^8 \cdot 5^9} = 2 \cdot 3^2 \cdot 5^3 \cdot \sqrt[3]{2^2 \cdot 3^2}$

10. To calculate the root of a root you multiply the indices.

    Example: $\sqrt[5]{\sqrt[3]{2}} = \sqrt[15]{2}$

11. To add or subtract radicals they must have the same index and the same radicand.

    Example: $5\sqrt[3]{2} + 3\sqrt[3]{2} - \sqrt[3]{2} = 7\sqrt[3]{2}$

12. Sometimes in algebra we want to find an equivalent expression for a radical expression that doesn't have any radicals in the denominator. This process is called ***rationalizing*** the denominator.

    The process consists of multiplying the numerator and the denominator by the same expression. There are three different cases:

    - Single square root

      Example:

      $\frac{2}{\sqrt{5}} = \frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{\sqrt{5^2}} = \frac{2\sqrt{5}}{5}$ You multiply numerator and denominator by $\sqrt{5}$.

- Single higher root

Example:

$$\frac{3}{\sqrt[3]{2}}=\frac{3}{\sqrt[3]{2}}\cdot\frac{\sqrt[3]{2^2}}{\sqrt[3]{2^2}}=\frac{3\sqrt[3]{2^2}}{\sqrt[3]{2^3}}=\frac{3\sqrt[3]{2^2}}{2}$$ You multiply numerator and denominator by $\sqrt[3]{2^2}$.

- Sums and differences of square roots. Multiply top and bottom by the difference, if the original is a sum and by the sum, if the original is a difference. This way you will get the *difference of squares* and get rid of the square roots.

Examples:

a) $$\frac{5}{1+\sqrt{2}}=\frac{5}{1+\sqrt{2}}\cdot\frac{1-\sqrt{2}}{1-\sqrt{2}}=\frac{5\cdot(1-\sqrt{2})}{(1+\sqrt{2})\cdot(1-\sqrt{2})}=\frac{5-5\sqrt{2}}{1^2-(\sqrt{2})^2}=\frac{5-5\sqrt{2}}{1-2}=\frac{5-5\sqrt{2}}{-1}=$$

$$=5\sqrt{2}-5$$

b) $$\frac{\sqrt{3}}{2-\sqrt{7}}=\frac{\sqrt{3}}{2-\sqrt{7}}\cdot\frac{2+\sqrt{7}}{2+\sqrt{7}}=\frac{\sqrt{3}\cdot(2+\sqrt{7})}{(2-\sqrt{7})\cdot(2+\sqrt{7})}=\frac{2\sqrt{3}+\sqrt{21}}{2^2-(\sqrt{7})^2}=\frac{2\sqrt{3}+\sqrt{21}}{4-7}=\frac{2\sqrt{3}+\sqrt{21}}{-3}$$

4. SCIENTIFIC NOTATION

Scientific notation (also known as standard form) is used for numbers that are either very large or very small, like:

- The mass of the Earth is about 6 000 000 000 000 000 000 000 000 kg
- X-rays have a wavelength of about 0,000 000 095 cm

Scientific notation is a more convenient way of expressing such numbers for working with calculators and computers.

A number in scientific notation is expressed as

$$N = a{,}bcdef\ldots \cdot 10^{n}$$

So,

- The mass of the Earth is about $6 \cdot 10^{24}$ kg
- X-rays have a wavelength of about $9{,}5 \cdot 10^{-8}$ cm

Let's see how to calculate with numbers expressed in scientific notation:

✓ Addition and subtraction: the numbers must be expressed with the same exponent.

Example:

$$2{,}13\cdot 10^2+5{,}36\cdot 10^4=2{,}13\cdot 10^2+536\cdot 10^2=538{,}13\cdot 10^2=5{,}3813\cdot 10^4$$

✓ Multiplication and division: you multiply or divide the decimal numbers and add or subtract the exponents.

Examples:

a) $(1{,}45\cdot 10^3)\cdot(2{,}06\cdot 10^5)=2{,}987\cdot 10^8$

b) $(3{,}21\cdot 10^7):(5{,}81\cdot 10^4)=0{,}5525\cdot 10^3=5{,}525\cdot 10^2$

PRONUNCIATION

- Diagonal |daɪ'ægənəl|
- Divine proportion |dɪ'vaɪn prə'pɔːʃən|
- Endpoint |'end'pɔɪnt|
- Fraction |'frækʃən|
- Golden section |'gəʊldən 'sekʃən|
- Indices |'ɪndɪsiːz|
- Infinity symbol |ɪn'fɪnɪtɪ 'sɪmbəl|
- Interval |'ɪntəvəl|
- Irrational |ɪ'ræʃənəl|
- Non-ending |nɒn 'endɪŋ|
- Parenthesis |pə'renθəsɪs|
- Phi |'faɪ|
- Prove-proof |pruːv pruːf|
- Radical |'rædɪkəl|
- Radicand |'rædɪˌkænd|
- Rational |'ræʃnəl|
- Rationalize |'ræʃnəlaɪz|
- Real |riːl|
- Root |ruːt|
- Scientific notation |ˌsaɪən'tɪfɪk nəʊ'teɪʃən|
- Set-builder |set 'bɪldə|
- Single |'sɪŋgəl|
- Thales’ theorem |'θeɪliːz 'θɪərəm|

WORKSHEET

1. Represent $\sqrt{17}$ and $\sqrt{34}$ on the number line.

2. Express these intervals using set-builder notation. Represent them on the number line:
   a) $[2, 5)=$
   b) $(-2, +\infty)=$
   c) $(-3, 8)=$
   d) $(-\infty, -6)=$

3. Simplify:
   a) $\sqrt[3]{2} \cdot \sqrt[3]{4}=$
   b) b) $\sqrt[4]{3} \cdot \sqrt[3]{5} =$
   c) $\frac{\sqrt[5]{a^4 \cdot b^3 \cdot c^6}}{\sqrt[4]{a \cdot b^2 \cdot c^3}}=$
   d) $\left(\sqrt[3]{x^2}\right)^9=$
   e) $\sqrt[3]{\sqrt{\sqrt[4]{2^{24}}}}=$
   f) $2\sqrt[3]{7} - \sqrt[3]{56} + 3\sqrt[3]{189} =$
   g) $\sqrt[3]{864a^4b^{10}} =$

4. Rationalize these expressions:
   a) $\frac{2}{\sqrt{11}} =$
   b) $\sqrt[3]{\frac{2}{3}} =$
   c) $\frac{3}{3-\sqrt{5}} =$
   d) $\frac{1}{\sqrt{2}+\sqrt{3}} =$

5. Calculate and express the result in scientific notation:
   a) $2{,}32 \cdot 10^5 - 1{,}47 \cdot 10^7 =$
   b) $(5{,}16 \cdot 10^4) \cdot (4{,}29 \cdot 10^5) =$
   c) $(3{,}67 \cdot 10^6) : (9{,}15 \cdot 10^2) =$
   d) $7{,}33 \cdot 10^9 - 8{,}09 \cdot 10^6 =$

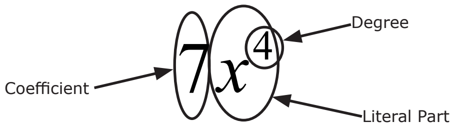
LESSON 2: POLYNOMIALS AND ALGEBRAIC FRACTIONS

Keywords

MONOMIAL coefficient DEGREE **leading term** Polynomial
quotient REMAINDER *binomial* **EVALUATE** Ruffini's Rule
FACTORING TRINOMIAL the simplest form THEOREM **algebraic**
fraction CANDIDATE *root* divisor

1. MONOMIALS

As you know, a **monomial** is an algebraic expression consisting of only one term, which has a known value (coefficient) multiplied by one or some unknown values represented by letters with exponents that must be constant and positive whole numbers (literal part). For example:



If the literal part of a monomial has only one letter, then the degree is the exponent of the letter. If the literal part of a monomial has more than one letter, then the degree is the addition of the exponents of the letters.

Examples:

The degree of $-5x^3$ is 3.

The degree of $-7x^2y^3$ is $2+3=5$.

Addition and subtraction of monomials

You can add monomials *only if they have the same literal part* (they are also called **like terms**). In this case, you *add the coefficients and leave the same literal part*.

Examples:

a) $3x+2x=5x$

b) $3x+2x^2$ (You cannot add the terms $3x$ and $2x^2$ because these are not like terms).

c) $5x^2+7-3x^2-4=2x^2+3$ (You cannot add the terms $2x^2$ and 3).

Multiplication of monomials

If you want to multiply two or more monomials, you just have to multiply the coefficients, and add the exponents of the equal letters:

Examples:

a) $2x^7 \cdot 3x^3 = 6x^{10}$ b) $\left(2xy^3\right)\cdot\left(-3xy^2\right) = -6x^2y^5$

Division of monomials

If you want to divide a monomial by a monomial of the same or lower degree, you just have to divide the coefficients, and subtract the exponents of the equal letters.

Examples:

a) $\left(10x^5\right)\div\left(-2x^2\right) = -5x^3$ b) $\left(12a^2b\right)\div\left(3a\right) = 4ab$

2. POLYNOMIALS

A **polynomial** is the addition or subtraction of two or more monomials (which are called terms).

-If there are two monomials, it is called a binomial, for example $x^2 + x$.

-If there are three monomials, it is called a trinomial, for example $3x^2 - 5x + 11$

The degree of the polynomial is the highest degree of the terms that it contains.

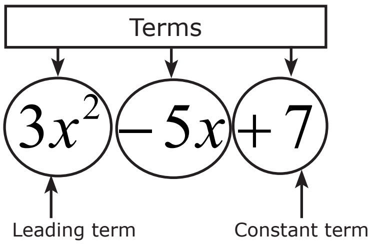
Examples:

a) $3x^2 - 5x + 11$ is a second-degree trinomial.

b) $x^4 - 5x$ is a fourth-degree binomial.

Example:

The following polynomial is a second-degree polynomial, and contains three terms: $3x^2$ is the leading term (the term with the highest exponent), and 7 is the constant term.



You usually write polynomials with the terms in "decreasing" order of exponents. We say that a polynomial is **complete** if it has terms of every exponent from the degree of the polynomial until zero.

Examples:

a) The polynomial $3x^2 - 5x + 7$ is complete.

b) The polynomial $3x^2 + 7$ is not complete.

Evaluating polynomials

"Evaluating" a polynomial $P(x)$ is calculating its numberical value at a given value of the variable: $x=a$. You must substitute the variable x for the value a, and calculate the value of the polynomial $P(a)$.

Example: Evaluate $P(x)=2x^3-x^2-4x+5$ at $x=-2$

Substitute x for -2 and calculate. But be careful with brackets and negative signs!

$$P(-2)=2\cdot(-2)^3-(-2)^2-4\cdot(-2)+5=2\cdot(-8)-4+8+5=-16-4+8+5=-20+13=-7$$

Addition of polynomials

When adding polynomials you must add each like term of the polynomial, that is, monomials that have the same literal part. (You must use what you know about the addition of monomials).

Example: Simplify $\left(3x^3-2x^2+5x-3\right)+\left(2x^3+4x^2-2x-4\right)$

$$\left(3x^3-2x^2+5x-3\right)+\left(2x^3+4x^2-2x-4\right)=5x^3+2x^2+3x-7$$

Subtraction of polynomials

When subtracting polynomials you must realise that a subtraction is the addition of the first term and the opposite of the second: A – B = A + (– B)

Example: Simplify: $\left(3x^3-2x^2+5x-3\right)-\left(2x^3+4x^2-2x-4\right)$

$$\left(3x^3-2x^2+5x-3\right)-\left(2x^3+4x^2-2x-4\right)=$$

$$3x^3-2x^2+5x-3-2x^3-4x^2+2x+4=x^3-6x^2+7x+1$$

Multiplication of polynomials

-**A Monomial times a multi-term polynomial.** To do this, we have to expand the brackets:

Example: Simplify $-2x(5x^2-x+10)$

$$-2x(5x^2-x+10)=-2x\cdot\left(5x^2\right)-2x\cdot\left(-x\right)-2x\cdot\left(10\right)=-10x^3+2x^2-20x$$

-**A Multi-term polynomial times a multi-term polynomial**. Look at this example:

Example: Simplify $\left(x+5\right)\cdot\left(x+2\right)$

$$\left(x+5\right)\cdot\left(x+2\right)=x\left(x\right)+x\left(2\right)+5\left(x\right)+5\left(2\right)=x^2+2x+5x+10=x^2+7x+10$$

Division of polynomials

You can divide a dividend polynomial P(x) by a divisor polynomial D(x) of the same or lower degree. The quotient polynomial Q(x) and the remainder polynomial R(x) (the degree of the remainder must be lower than the degree of the divisor) can be determined as follows:

Example: Divide the polynomial $P(x)=6x^4-5x^2+7x-10$ by $D(x)=2x^2+4x-1$

-Step 1: If the dividend polynomial P(x) is not complete, leave gaps for the missing terms (this will allow you to line up like terms later).

In the example, place the terms like this: $6x^4 \qquad -5x^2+7x-10 \;\; \left| \underline{\,2x^2+4x-1\,} \right.$

-Step 2: Divide the leading term of the dividend polynomial P(x) by the leading term of the divisor polynomial D(x) to get the leading term of the quotient polynomial.

In the example, $\left(6x^4\right)\div\left(2x^2\right)=3x^2$, so you have:

$$\begin{array}{lll} 6x^4 & \qquad -5x^2+7x-10 & \left| \underline{\,2x^2+4x-1\,} \right. \\ & & \;\; 3x^2 \end{array}$$

-Step 3: Take the term found in Step 2 and multiply it by the divisor polynomial D(x). The opposite terms of this product must be lined up with the terms of the dividend of the same degree (like terms):

$$\begin{array}{lll} \quad\;\; 6x^4 & \qquad -5x^2+7x-10 & \left| \underline{\,2x^2+4x-1\,} \right. \\ -6x^4-12x^3+3x^2 & & \;\; 3x^2 \end{array}$$

-Step 4: Add every term found in Step 3 with the like term above. The result is the first remainder polynomial.

$$\begin{array}{lll} \quad\;\; 6x^4 & \qquad -5x^2+7x-10 & \left| \underline{\,2x^2+4x-1\,} \right. \\ \underline{-6x^4-12x^3+3x^2} & & \;\; 3x^2 \\ \qquad -12x^3-2x^2 & & \end{array}$$

Repeat the last three steps until the degree of the final remainder polynomial R(x) is lower than the degree of the divisor polynomial D(x). In Step 2 the following terms of the quotient will be obtained dividing the leading term of the remainder polynomial by the leading term of the divisor polynomial D(x)).

In the example, if you repeat these steps two more times, you finally get:

$$\begin{array}{lll} \quad\;\; 6x^4 & \qquad -5x^2+7x-10 & \left| \underline{\,2x^2+4x-1\,} \right. \\ \underline{-6x^4-12x^3+3x^2} & & \;\; 3x^2-6x+11 \longleftarrow \text{Quotient} \\ \qquad -12x^3-2x^2 & & \\ \qquad \underline{12x^3+24x^2-6x} & & \\ \qquad\qquad 22x^2+x & & \\ \qquad\quad \underline{-22x^2-44x+11} & & \\ \qquad\qquad\quad -43x+1 \longleftarrow \text{Remainder} & & \end{array}$$

You can check that the division is correct verifying the property:

$$P(x)=D(x)\cdot Q(x)+R(x)$$

(Dividend$=$Divisor$\cdot$Quotient$+$Remainder)

Note: If the remainder polynomial is zero ($R(x)=0$), then you have $P(x)=D(x)\cdot Q(x)$ and you say that $P(x)$ is divisible by $D(x)$.

Ruffini's rule: division of a polynomial by a binomial of the form *x-a*

To apply Ruffini's rule you can follow the steps of this example:

Example: Divide the polynomial $x^4-x^2-7x+10$ by $x-2$

-Step 1: Set the coefficients of the dividend in one line. If the polynomial is not complete, complete it by adding the missing terms with zeroes. Draw two perpendicular lines like this:

$$\begin{array}{r|rrrrr} & 1 & 0 & -1 & -7 & 10 \\ & & & & & \\ \hline & & & & & \end{array}$$

-Step 2: At the bottom left, place the opposite of the independent term of the divisor.

$$\begin{array}{r|rrrrr} & 1 & 0 & -1 & -7 & 10 \\ 2 & & & & & \\ \hline & & & & & \end{array}$$

-Step 3: Bring down the first coefficient.

$$\begin{array}{r|rrrrr} & 1 & 0 & -1 & -7 & 10 \\ 2 & & & & & \\ \hline & 1 & & & & \end{array}$$

-Step 4: Multiply this coefficient by the divisor and place it under the following coefficient.

$$\begin{array}{r|rrrrr} & 1 & 0 & -1 & -7 & 10 \\ 2 & & 2 & & & \\ \hline & 1 & & & & \end{array}$$

-Step 5: Add the two coefficients.

$$\begin{array}{r|rrrrr} & 1 & 0 & -1 & -7 & 10 \\ 2 & & 2 & & & \\ \hline & 1 & 2 & & & \end{array}$$

Repeat Steps 4 and 5 until you get the last number, like this:

$$\begin{array}{r|rrrrr} & 1 & 0 & -1 & -7 & 10 \\ 2 & & 2 & 4 & 6 & -2 \\ \hline & 1 & 2 & 3 & -1 & \boxed{8} \leftarrow \text{Remainder} \\ & \uparrow & \uparrow & \uparrow & \uparrow & \end{array}$$

Coefficients of the Quotient

The last number obtained, 8, is the remainder of the division.

The quotient is a polynomial of one degree less than the dividend polynomial and whose coefficients are the ones obtained in the division.

In this example, the quotient polynomial is: $Q(x) = x^3 + 2x^2 + 3x - 1$

3. APPLICATIONS OF RUFFINI'S RULE

3.1. Evaluating polynomials

The following theorem is useful for evaluating polynomials at a given value of x:

THE REMAINDER THEOREM

The value of a polynomial $P(x)$ at a given value of the variable $x = a$ coincides with the remainder of the division $P(x) \div (x-a)$.

Because of this theorem, you can use Ruffini's Rule to evaluate polynomials. Let's see an example:

Example: Evaluate $P(x) = 2x^3 - x^2 - 4x + 5$ at $x = -2$

Let's apply Ruffini's Rule:

$$\begin{array}{r|rrrr} & 2 & -1 & -4 & 5 \\ -2 & & -4 & 10 & -12 \\ \hline & 2 & -5 & 6 & \boxed{-7} \end{array}$$

So, you have that $P(-2) = -7$

3.2. Divisibility criteria of a polynomial by a binomial of the form ***x-a***

given a polynomial $P(x)$ of integer coefficients and given an integer number $a \in \mathbb{Z}$, if $P(x) \div (x-a)$ is an exact division (with zero as remainder), Ruffini's Rule states that a must be an integer divisor of the constant term of $P(x)$.

Example: Is the polynomial $P(x) = 2x^3 - 2x^2 - 10x - 6$ divisible by the binomial $x-5$?
The answer is "No", because the number 5 is not an integer divisor of the constant term -6.

Example: Is the polynomial $P(x) = 2x^3 - 2x^2 - 10x - 6$ divisible by the binomial $x-1$?
The integer divisors of the constant term of this polynomial are ± 1, ± 2, ± 3 and

± 6. As 1 is included on this list, the division $\left(2x^3-2x^2-10x-6\right):\left(x-1\right)$ could be exact. Let’s check if the remainder is zero:

$$\begin{array}{r|rrrr} & 2 & -2 & -10 & -6 \\ 1 & & 2 & 0 & -10 \\ \hline & 2 & 0 & -10 & \boxed{-16} \end{array} \longleftarrow \text{Remainder}$$

As the remainder is not zero, you can conclude that the polynomial $P(x)=2x^3-2x^2-10x-6$ is not divisible by $x-1$.

Example: Is the polynomial $P(x)=2x^3-2x^2-10x-6$ divisible by the binomial $x-3$?

The integer divisors of the constant term of this polynomial are ± 1, ± 2, ± 3 and ± 6. As 3 is included on this list, the division $\left(2x^3-2x^2-10x-6\right):\left(x-3\right)$ could be exact. Let’s check if the remainder is zero:

$$\begin{array}{r|rrrr} & 2 & -2 & -10 & -6 \\ 3 & & 6 & 12 & 6 \\ \hline & 2 & 4 & 2 & \boxed{0} \end{array} \longleftarrow \text{Remainder}$$

As the remainder is zero, you can conclude that the polynomial $P(x)=2x^3-2x^2-10x-6$ is divisible by $x-3$.

4. FACTORING POLYNOMIALS

-A real number a is called **root** of a polynomial $P(x)$ if follows that $P(a)=0$. So the roots of a polynomial are the solutions of the equation $P(x)=0$.

-When you are searching for the roots of a polynomial $P(x)$ you can try with the integer divisors of its constant term. Once you find a root a, as $P(x)$ is divisible by $x-a$, you can rewrite $P(x)=(x-a)\cdot Q(x)$ and continue searching for the roots of $P(x)$ in the polynomial $Q(x)$.

-**Factoring a polynomial** means rewriting it as a product of polynomials of the lowest degree as possible that can be multiplied together to give us the polynomial that you started with.

Let’s see some examples:

Example 1: Factor the polynomial $3x^2+6x$

In this example each term has a common factor $3x$. So, you can **extract the common factor**:

$$3x^2+6x=3x\cdot(x+2)$$

Example 2: Factor the polynomial x^2-25

In this example you can use the **special product** $(A+B)\cdot(A-B)=A^2-B^2$ to factor the polynomial. So, you can rewrite:

$$x^2-25=\left(x+5\right)\cdot\left(x-5\right)$$

Example 3: Factor the polynomial x^2+6x+9

In this example you can use the **special product** $(A+B)^2 = A^2 + 2\cdot A\cdot B + B^2$ to factor the polynomial. So, you can rewrite:

$$x^2 + 6x + 9 = x^2 + 2\cdot 3x + 3^2 = (x+3)^2$$

Example 4: Factor the polynomial $x^2 - 6x + 9$

In this example you can use the **special product** $(A-B)^2 = A^2 - 2\cdot A\cdot B + B^2$ to factor the polynomial. So, you can rewrite:

$$x^2 - 6x + 9 = x^2 - 2\cdot 3x + 3^2 = (x-3)^2$$

Example 5: Factor the polynomial $x^2 + x - 6$
In this example you can't use any special product and there isn't a common factor to every term. So, you can search for the roots of this polynomial, which are the solutions of the equation $x^2 + x - 6 = 0$

Using the quadratic formula, you get that the roots are:

$$x^2 + x - 6 = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1+24}}{2} = \frac{-1 \pm 5}{2} = \left\langle \begin{array}{l} 5 \\ -3 \end{array} \right.$$

So, you can rewrite: $x^2 + x - 6 = (x-2)\cdot(x+3)$

Example 6: Factor the polynomial $2x^2 - 11x + 5$

If you use the quadratic formula, you get that the roots of the polynomial are:

$$2x^2 - 11x + 5 = 0 \Rightarrow x = \frac{11 \pm \sqrt{121-40}}{4} = \frac{11 \pm 9}{4} = \left\langle \begin{array}{l} 5 \\ \frac{1}{2} \end{array} \right.$$

So, the factored polynomial is: $2x^2 - 11x + 5 = 2\cdot(x-5)\cdot\left(x - \frac{1}{2}\right)$

Notice that if you factor a polynomial, you mustn't forget to multiply by its leading coefficient

Example 7: Factor the polynomial $x^3 - 2x^2 - 5x + 6$
Since the degree of the polynomial is 3 and you can't extract common factor, you can apply Ruffini's Rule in order to search for the integer roots of the polynomial. Our candidates are the divisors of the constant term 6, which are: $\pm 1, \pm 2, \pm 3$ and ± 6.

If you try with 1, you get:

$$\begin{array}{c|cccc}
 & 1 & -2 & -5 & 6 \\
1 & & 1 & -1 & -6 \\
\hline
 & 1 & -1 & -6 & \boxed{0}
\end{array}$$

As the remainder is zero, you get that 1 is a root of the polynomial, and $x-1$ is a factor. So, you can rewrite: $x^3-2x^2-5x+6=(x-1)\cdot(x^2-x-6)$

Now you can continue searching for the roots of the polynomial using the quadratic formula, and you get:

$$x^2-x-6=0 \Rightarrow x=\frac{1\pm\sqrt{1+24}}{2}=\frac{1\pm 5}{2}=\left\langle\begin{matrix}3\\-2\end{matrix}\right.$$

You finally get the factored polynomial:

$$x^3-2x^2-5x+6=(x-1)\cdot(x-3)\cdot(x+2)$$

Note: As all the roots of this polynomial are integer numbers you could also have found them using Ruffini's Rule three times.

$$\begin{array}{c|cccc}
 & 1 & -2 & -5 & 6 \\
1 & & 1 & -1 & -6 \\
\hline
 & 1 & -1 & -6 & \boxed{0} \\
-2 & & -2 & 6 & \\
\hline
 & 1 & -3 & \boxed{0} & \\
3 & & 3 & & \\
\hline
 & 1 & \boxed{0} & &
\end{array}$$

Example 8: Factor the polynomial $x^6-6x^5+9x^4+4x^3-12x^2$

First of all, you can extract common factor x^2 and you can rewrite:

$$x^6-6x^5+9x^4+4x^3-12x^2=x^2\cdot\left(x^4-6x^3+9x^2+4x-12\right)$$

Now you can continue factoring by searching for the roots of the polynomial $x^4-6x^3+9x^2+4x-12$. As it is a fourth-degree polynomial you can use Ruffini's rule and try with the divisors of its constant term -12, which are: $\pm 1,\ \pm 2,\ \pm 3,\ \pm 4,\ \pm 6$ and ± 12.

$$\begin{array}{c|ccccc}
 & 1 & -6 & 9 & 4 & -12 \\
-1 & & -1 & 7 & -16 & 12 \\
\hline
 & 1 & -7 & 16 & -12 & \boxed{0} \\
2 & & 2 & -10 & 12 & \\
\hline
 & 1 & -5 & 6 & \boxed{0} &
\end{array}$$

Then, you have that $x^6-6x^5+9x^4+4x^3-12x^2=x^2\cdot\left(x+1\right)\cdot\left(x-2\right)\cdot\left(x^2-5x+6\right)$

Finally, you can search for roots of the polynomial x^2-5x+6 using the quadratic formula:

$$x^2-5x+6=0 \Rightarrow x=\frac{5\pm\sqrt{25-24}}{2}=\frac{5\pm1}{2}=\left\langle\begin{matrix}3\\2\end{matrix}\right.$$

So, the original polynomial can be factored like this:

$$x^6-6x^5+9x^4+4x^3-12x^2=x^2\cdot(x+1)\cdot(x-2)^2\cdot(x-3)$$

<u>Note</u>: The roots of the original polynomial are 0 (double), -1, 2 (double) and 3.

<u>Example 9</u>: Factor the polynomial x^3-4x^2+4x-3

You can use Ruffini's Rule to try to find the integer roots of the polynomial. The candidates (divisors of the constant term) are ±1 and ±3. Luckily, if you try with number 3, you get remainder zero.

$$\begin{array}{c|cccc} & 1 & -4 & 4 & -3 \\ 3 & & 3 & -3 & 3 \\ \hline & 1 & -1 & 1 & \boxed{0} \end{array}$$

So you can rewrite: $x^3-4x^2+4x-3=(x-3)\cdot(x^2-x+1)$

Now you can use the quadratic formula to find the roots of x^2-x+1

$$x^2-x+1=0 \Rightarrow x=\frac{1\pm\sqrt{1-4}}{2}=\frac{1\pm\sqrt{-3}}{2}\notin\mathbb{R}$$

So there aren't more real roots of the given polynomial and you cannot factor anymore.

5. <u>DIVISIBILITY OF POLYNOMIALS</u>

-A polynomial $Q(x)$ is a **factor** or **divisor** of a polynomial $P(x)$ if the division $P(x)\div Q(x)$ is exact (with zero as remainder). In that case, you can also say that $P(x)$ is a multiple of $Q(x)$.

-A polynomial $P(x)$ is said to be **in the simplest form** if it hasn't got any factor of lower degree than $P(x)$.

-As you usually do with natural numbers, you can calculate the **Highest Common Factor** (**H.C.F.**) and the **Lowest Common Multiple** (**L.C.M.**) of two or more polynomials.

<u>Example</u>: Calculate the H.C.F. and the L.C.M. of the following polynomials:

$P(x)=x^6-6x^5+9x^4+4x^3-12x^2$ and $Q(x)=x^3-2x^2-5x+6$

These polynomials appeared in some of the last examples, so you know that you can factor them as follows:

$$P(x)=x^6-6x^5+9x^4+4x^3-12x^2=x^2\cdot(x+1)\cdot(x-2)^2\cdot(x-3)$$
$$Q(x)=x^3-2x^2-5x+6=(x-1)\cdot(x-3)\cdot(x+2)$$

To calculate the H.C.F. of these polynomials you multiply the common factors and power each one of them to the lowest exponent:

$$H.C.F.\left(P(x),Q(x)\right)=x-3$$

To calculate the L.C.M. of these polynomials you multiply all the factors and power each one of them to the highest exponent:

$$L.C.M.\left(P(x),Q(x)\right)=x^2\cdot(x+1)\cdot(x-1)\cdot(x+2)\cdot(x-2)^2\cdot(x-3)$$

6. ALGEBRAIC FRACTIONS

An **algebraic fraction** is the quotient of two polynomials in the form $\frac{P(x)}{Q(x)}$.

-As you usually do with numerical fractions, you can **simplify algebraic fractions** factoring the polynomials in the numerator and in the denominator. Dividing by the H.C.F. of numerator and denominator you will get **the simplest form of the algebraic fraction**.

Example: Simplify: $\frac{x^6-6x^5+9x^4+4x^3-12x^2}{x^3-2x^2-5x+6}\underset{\uparrow}{=}\frac{x^2\cdot(x+1)\cdot(x-2)^2\cdot(x-3)}{(x-1)\cdot(x-3)\cdot(x+2)}=\frac{x^2\cdot(x+1)\cdot(x-2)^2}{(x-1)\cdot(x+2)}$

See the last example

As you usually do with numerical fractions, you can also **add, subtract, multiply or divide algebraic fractions**. (To add or subtract algebraic fractions you need to reduce to common denominator).

Examples:

-Add: $\frac{2}{x}+\frac{x}{x-1}=\frac{2(x-1)}{x\cdot(x-1)}+\frac{x^2}{x\cdot(x-1)}=\frac{x^2+2x-2}{x\cdot(x-1)}$

-Subtract: $\frac{2}{x}-\frac{x}{x-1}=\frac{2(x-1)}{x\cdot(x-1)}-\frac{x^2}{x\cdot(x-1)}=\frac{-x^2+2x-2}{x\cdot(x-1)}$

-Multiply: $\frac{2x}{x-2}\cdot\frac{3x-6}{2\cdot(x+1)}=\frac{2x(3x-6)}{(x-2)\cdot2\cdot(x+1)}=\frac{x\cdot3\cdot(x-2)}{(x-2)\cdot(x+1)}=\frac{3x}{x+1}$

-Divide: $\frac{x}{x-1}\div\frac{x^2}{x^2-1}=\frac{x\cdot(x^2-1)}{(x-1)\cdot x^2}=\frac{x\cdot(x+1)\cdot(x-1)}{(x-1)\cdot x^2}=\frac{x+1}{x}$

PRONUNCIATION

- Algebraic |ˌældʒɪ'breɪk|
- Binomial |baɪ'nəʊmɪəl|
- Candidate |'kændɪdət|
- Coefficient |ˌkəʊɪ'fɪʃnt|
- Degree |dɪ'griː|
- Divisor |dɪ'vaɪzə|
- Evaluate |ɪ'væljʊeɪt|
- Factoring |'fæktəɪŋ|
- Leading term |'liːdɪŋ tɜːm|
- Monomial |mɒ'nəʊmɪəl|
- Polynomial |ˌpɒlɪ'nəʊmɪəl|
- Quadratic |kwɒ'drætɪk|
- Quotient |'kwəʊʃnt|
- Remainder |rɪ'meɪndə|
- Root |ruːt|
- Ruffini’s Rule |ruː'fiːniz ruːl|
- The simplest form |ðə 'sɪmplɪst fɔːm|
- Theorem |'θɪərəm|
- Trinomial |traɪ'nəʊmɪəl|

WORKSHEET

1. Divide the following polynomials. (You must indicate the quotient polynomial and the remainder polynomial).
   a) $\left(x^5-3x^4-11x^3+16x^2-11x+6\right)\div\left(x^2+2x-3\right)$
   b) $\left(x^5-2x^4-x^3+8x-2\right)\div\left(x^3-2x-5\right)$
   c) $\left(3x^4+7x^3-x^2+12x-4\right)\div\left(x+3\right)$
   d) $\left(3x^6-6x^5-3x^3-x^2+16x-12\right)\div\left(x-2\right)$

2. a) Apply Ruffini's Rule to evaluate the polynomial $P(x)=3x^5-12x^2+3x-5$ at $x=-2$

   b) What is the remainder of the division $\left(3x^5-12x^2+3x-5\right)\div\left(x+2\right)$? Explain your answer.

3. Factor these polynomials:
   a) $x^7-6x^6+13x^5-12x^4+4x^3\ x^7-6x^6+13x^5-12x^4+4x^3$
   b) $4x^3-8x^2+x+3$

4. A safe box has a four-digit security code that consists of the roots of the polynomial $x^4-8x^3+17x^2-10x$ in decreasing order. What is the security code?

5. The polynomial $P(x)=x^4+mx^3+16x^2-25$ is divisible by x+5. What is the value of m?

6. The remainder of the division $\left(x^4-x^2+mx+3\right)\div\left(x-2\right)$ is 5. What is the value of m?

7. Given the polynomial $P(x)=x^3-2x^2+mx+n$. We know that the remainder of the division $P(x)\div\left(x-2\right)$ is -4. We also know that 1 is a root of $P(x)$· Calculate the values of m and n.

8. Calculate the Highest Common Factor and the Lowest Common Multiple of the polynomials $P(x)=x^6+x^5-7x^4-13x^3-6x^2$ and $Q(x)=x^6+6x^5+11x^4+6x^3$

9. Simplify the following algebraic fractions:
   a) $\dfrac{2x^2-2}{6x^2+12x+6}$
   b) $\dfrac{x^6+x^5-7x^4-13x^3-6x^2}{x^6+6x^5+11x^4+6x^3}$